1. (21 points) Calculate:

(a) The length of the projection of the vector $\mathbf{A} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ onto the vector $\mathbf{B} = -2\mathbf{i} - \mathbf{j} + \mathbf{k}$.

$$\text{length} = \left|\text{comp}_{\vec{B}} \vec{A}\right| = \left|\frac{\vec{A} \cdot \vec{B}}{|\vec{B}|}\right|$$

$$= \frac{|-5|}{\sqrt{4+1+1}} = \frac{5}{\sqrt{6}}$$

(b) A vector equation for the line that contains the point $(2, -1, 3)$ and is perpendicular to the plane $x + 3y - 2z = 4$.

dir vector for line $\vec{D} = \vec{i} + 3\vec{j} - 2\vec{k}$

$$\vec{L}(t) = (2+t)\vec{i} + (-1+3t)\vec{j} + (3-2t)\vec{k}$$

(c) The area of the triangle in space with vertices $P = (0, 1, 0)$, $Q = (4, 1, -2)$, and $R = (5, 3, 1)$.

$\vec{PQ} = 4\vec{i} - 2\vec{k}$ $\quad$ $\vec{PR} = 5\vec{i} + 2\vec{j} + \vec{k}$

area triangle $= \frac{1}{2}|\vec{PQ} \times \vec{PR}|$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 0 & -2 \\ 5 & 2 & 1 \end{vmatrix} = 4\vec{i} - 14\vec{j} + 8\vec{k}$$

$$\text{area} = \frac{1}{2}\sqrt{16 + 196 + 64} = \frac{\sqrt{276}}{2}$$

2. (18 points) Let

$$\mathbf{r}(t) = (3t^2)\,\mathbf{i} + (te^t)\,\mathbf{j} + (\ln t)\,\mathbf{k},$$

and suppose that $\mathbf{r}(t)$ is the position function of a particle moving through space. Calculate the velocity, acceleration, and speed of the particle when it is at the point $(3, e, 0)$.

$\vec{r}(t) = 3\vec{i} + e\vec{j} \iff \left.\begin{array}{l} 3t^2 = 3 \checkmark \\ te^t = e \checkmark \end{array}\right\}$ for $t=1$

$\ln t = 0 \Rightarrow t = 1$

$\vec{r}\,'(t) = (6t)\vec{i} + (te^t + e^t)\vec{j} + \left(\frac{1}{t}\right)\vec{k}$

$= (6t)\vec{i} + (t+1)e^t\vec{j} + \left(\frac{1}{t}\right)\vec{k}$

$\vec{r}\,''(t) = 6\vec{i} + \underbrace{(e^t + (t+1)e^t)}_{(t+2)e^t}\vec{j} + \left(-\frac{1}{t^2}\right)\vec{k}$

Velocity at $(3, e, 0)$:

$\vec{r}\,'(1) = 6\vec{i} + (2e)\vec{j} + \vec{k}$

acceleration at $(3, e, 0)$:

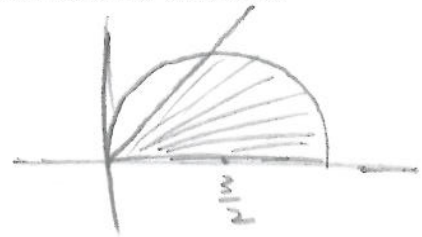
$\vec{r}\,''(1) = 6\vec{i} + 3e\vec{j} - \vec{k}$

speed at $(3, e, 0) = |\vec{r}\,'(1)|$

$= \sqrt{36 + 4e^2 + 1}$

$= \sqrt{37 + 4e^2}$

3. (20 points) Find the area of the region in the first quadrant that is inside the circle $r = 3\cos\theta$ and below the line $y = x$. (The half-angle formula for cosine is $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$.)




$y = x$ first quadrant

$\iff \theta = \pi/4$

$$\text{area} = \frac{1}{2}\int_0^{\pi/4} r^2\, d\theta = \frac{1}{2}\int_0^{\pi/4} (3\cos\theta)^2\, d\theta$$

$$= \frac{9}{2}\int_0^{\pi/4} \cos^2\theta\, d\theta$$

$$= \frac{9}{4}\int_0^{\pi/4} (1 + \cos 2\theta)\, d\theta$$

Let $u = 2\theta \implies du = 2\, d\theta$

$\theta = 0 \implies u = 0$

$\theta = \frac{\pi}{4} \implies u = \frac{\pi}{2}$

$$\text{area} = \frac{9}{4}\int_0^{\pi/2} (1 + \cos u)\left(\frac{1}{2}\right) du$$

$$= \frac{9}{8}\Big[u + \sin u\Big]_0^{\pi/2}$$

$$= \frac{9}{8}\left[\frac{\pi}{2} + 1\right]$$

4. (18 points) The two lines

$$\mathbf{L}_1(t) = (1+t)\,\mathbf{i} + (1-2t)\,\mathbf{j} + (3t)\,\mathbf{k}$$

and

$$\mathbf{L}_2(t) = (2+t)\,\mathbf{i} + (11+2t)\,\mathbf{j} + (-3+t)\,\mathbf{k}$$

intersect.

(a) Where do they intersect? Use $s$ as the parameter for $L_2$.

$$\begin{cases} 1+t = 2+s \Rightarrow t = s+1 \\ 1-2t = 11+2s \Rightarrow 1-2(s+1) = 11+2s \Rightarrow 4s = -12 \\ 3t = -3+t \quad (\text{check: } -6 \overset{\checkmark}{=} -6) \end{cases} \Rightarrow s = -3, \quad t = -2$$

$\vec{L}_1(-2) = -\vec{i} + 5\vec{j} - 6\vec{k}$

$(-1, 5, -6)$

(b) Calculate the angle of intersection.

$\vec{D}_1 = \text{dir vector } \vec{L}_1 = \vec{i} - 2\vec{j} + 3\vec{k}$

$\vec{D}_2 = \text{dir vector } \vec{L}_2 = \vec{i} + 2\vec{j} + \vec{k}$

$$\cos\theta = \frac{\vec{D}_1 \cdot \vec{D}_2}{|\vec{D}_1||\vec{D}_2|} = \frac{0}{|\vec{D}_1||\vec{D}_2|} = 0 \qquad \theta = \pi/2 \text{ radians}$$

(c) Find an equation for the plane that contains these two lines.

$\vec{N} = \text{normal vec} = \vec{D}_1 \times \vec{D}_2$

$$\vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 3 \\ 1 & 2 & 1 \end{vmatrix} = -8\vec{i} + 2\vec{j} + 4\vec{k}$$

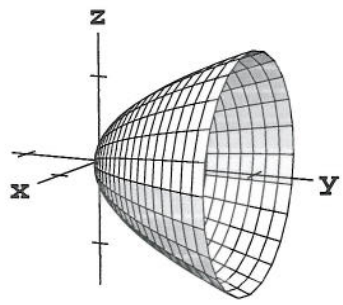
$\vec{N} \cdot \vec{PQ} = 0 \Rightarrow -8(x+1) + 2(y-5) + 4(z+6) = 0$

$-8x + 2y + 4z = -6$ or

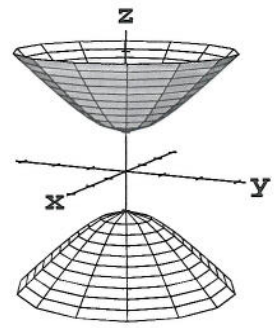
$-4x + y + 2z = -3.$

5. (20 points) Here are two surfaces in space:

**Surface A**




**Surface B**




Here are 6 equations of surfaces:

$z^2 = x^2 + y^2 + 1 \longrightarrow$ 1. $z^2 - x^2 - y^2 = 1$ 2. $x^2 - y^2 + z^2 = 0 \longleftarrow y^2 = x^2 + z^2$

plane $\longrightarrow$ 3. $2x + y - z = 3$ 4. $x^2 - y + z^2 = 0 \longleftarrow y = x^2 + z^2$

$z = x^2 + y^2 \longrightarrow$ 5. $x^2 + y^2 - z = 0$ 6. $x^2 + y^2 - z^2 = 1 \longleftarrow x^2 + y^2 = 1 + z^2$

For each surface, pick the equation that describes it. Provide a brief justification for your choice. **You will not receive any credit unless you provide a valid justification.**

(a) The equation for surface A is 4. My reason for choosing this answer is:

The equation for #4 is $y = x^2 + z^2$. Therefore, $y$ is never negative, and slices by planes $y = k > 0$ are circles. The $x = 0$ slice is the parabola $y = z^2$, and the $z = 0$ is the parabola $y = x^2$.

(b) The equation for surface B is 1. My reason for choosing this answer is:

From $z^2 = x^2 + y^2 + 1$, we see that either $z \geq 1$ or $z \leq -1$. The horizontal slices $z = k$ are circles of radius $\sqrt{k^2 - 1}$. The $x = 0$ slice is the hyperbola $z^2 - y^2 = 1$, and the $y = 0$ slice is the hyperbola $z^2 - x^2 = 1$.