1. (12 points) Calculate:

(a) the gradient of the function $f(x, y) = e^{xy} \sin 2x$ at the point $(\pi/6, 0)$.
\[
\frac{\partial f}{\partial x} = ye^{xy} \sin 2x + e^{xy} (\cos 2x y x 2x)
\]
\[
\frac{\partial f}{\partial y} = xe^{xy} \sin 2x
\]
\[
\nabla f(\pi/6, 0) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} 
\]
\[
= \frac{\pi}{6} + \left( \frac{\sqrt{3}}{2} \right) \frac{\pi}{6}
\]

(b) the projection $\text{proj}_b a$ of the vector $a = i + 4j - k$ in the direction of $b = 2i - 3j + 2k$.
\[
\text{proj}_b a = \frac{a \cdot b}{|b|^2} (b) = \left( \frac{a \cdot b}{b \cdot b} \right) b
\]
\[
= \frac{-12}{17} \left( 2t - 3 \frac{t}{7} + 2z \right)
\]
\[
= -\frac{24}{17} t + \frac{36}{17} \frac{t}{7} - \frac{24}{17} z
\]

(c) the curl of the vector field $F(x, y, z) = x^3y^2z^2i + 3x^3y^2z^2j + 2x^3y^2z^2k$.
\[
\text{curl } F = \nabla \times F = \begin{vmatrix}
\frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\
x^3y^2z^2 & 3x^3y^2z^2 & 2x^3y^2z^2 \\
x^3y^2z^2 & 3x^3y^2z^2 & 2x^3y^2z^2
\end{vmatrix}
\]
\[
= (2x^2z^2 - 6x^3y^2) \frac{t}{2}
\]
\[
- (4xy^2z^2 - 2x^3y^2) \frac{t}{2}
\]
\[
+ (9x^2y^2z^2 - x^3z^2) \frac{t}{2}
\]
2. (12 points) Which of the following five lines are parallel? Which are equal? **In order to receive any credit for your answer, you must provide brief justifications for your assertions.**

\[ \ell_1 : x = 1 + t, \quad y = t, \quad z = 2 - 5t \quad (1,0,2) \]
\[ \ell_2 : x + 1 = y - 2 = 1 - z \quad (-1,2,1) \]
\[ \ell_3 : x = 1 + t, \quad y = 4 + t, \quad z = 1 - t \]
\[ \ell_4 : r(t) = (2 + 2t)i + (1 + 2t)j - (3 + 10t)k \quad (2,1,-3) \]
\[ \ell_5 : x = 2 + t, \quad y = 3 + t, \quad z = 2 + t \]

**Direction Vectors:**
\[ \vec{D}_1 = \mathbf{i} + \mathbf{j} - 5\mathbf{k} \]
\[ \vec{D}_2 = \mathbf{i} + \mathbf{j} - 2\mathbf{k} \]
\[ \vec{D}_3 = \mathbf{i} + \mathbf{j} - \mathbf{k} \]
\[ \vec{D}_4 = 2\mathbf{i} + 2\mathbf{j} - 10\mathbf{k} \]
\[ \vec{D}_5 = \mathbf{i} + \mathbf{j} + 2\mathbf{k} \]

\( \vec{D}_1 \) and \( \vec{D}_4 \) are parallel
\( \vec{D}_2 \) and \( \vec{D}_3 \) are parallel

The point \((2,1,-3)\) satisfies \( \ell_1 \) and \( \ell_4 \), so those lines are equal.

The point \((-1,2,1)\) lies on \( \ell_2 \) but not on \( \ell_3 \), so \( \ell_2 \) and \( \ell_3 \) are parallel.

\( \ell_5 \) is not parallel to any of the other four lines.
3. (12 points) Consider the following region \( R \) in the \( xy \)-plane. It is bounded by the curves \( y = x^2 \) and \( x = y^2 + 1 \) and the lines \( y = -1 \), \( y = 1 \), and \( x = -1 \). Calculate

\[
\iint_R 6xy^2 \, dA.
\]

\[
\iint_R 6xy^2 \, dA = \iint_{R_1} 6xy^2 \, dA + \iint_{R_2} 6xy^2 \, dA
\]

\[
\iint_{R_1} 6xy^2 \, dA = \int_{-1}^{1} \int_{-1}^{x} 6xy^2 \, dy \, dx
\]

\[
= \int_{-1}^{1} 2x \left[ \frac{y^3}{3} \right]_{y=-1}^{y=x} \, dx = \int_{-1}^{1} 2x(x^6 + 1) \, dx
\]

\[
= \left[ \frac{x^8}{4} + x^2 \right]_{x=-1}^{x=1} = (1 + 1) - (-1 + 1) = 0
\]

\[
\iint_{R_2} 6xy^2 \, dA = \int_{-1}^{1} \int_{1}^{y^2 + 1} 6xy^2 \, dx \, dy
\]

\[
= \int_{-1}^{1} 3y^2 \left[ x \right]_{x=1}^{x=y^2 + 1} \, dy = \int_{-1}^{1} 3y^2(y^4 + 2y^2 + 1 - 1) \, dy
\]

\[
= \int_{-1}^{1} 3y^2 + 6y^4 \, dy = \left[ \frac{3}{5} y^5 + \frac{6}{5} y^5 \right]_{y=-1}^{y=1}
\]

\[
= \left( \frac{3}{5} + \frac{6}{5} \right) - \left( -\frac{3}{5} - \frac{6}{5} \right) = \frac{57}{35} + \frac{57}{35}
\]

\[
= \frac{114}{35}
\]
4. (12 points) Consider the curves

\[ r_1(t) = e^t i + tj + (t - 1)k \quad \text{and} \quad r_2(t) = ti + (t - 1)j - t^2k. \]

Determine their point(s) of intersection and the angle(s) at which they intersect.

Write 2nd curve as

\[ r_2(s) = s \bar{r} + (s-1) \bar{r} - s^2 \bar{r} \]

\[ s = e^t \]

\[ s-1 = t \quad \Rightarrow \quad s-2 = -s^2 \to s^2 + s - 2 = 0 \]

\[ (s+2)(s-1) = 0 \]

\[ s = -2 \text{ or } s = 1 \]

\[ s = -2 \Rightarrow t = -3, \text{ but } -2 \neq e^{-3} \Rightarrow \text{ not a point of intersection} \]

\[ s = 1 \Rightarrow t = 0 \text{ and } e^0 = 1 \Rightarrow \text{ intersection point of intersection is } (1, 0, -1) \]

\[ \bar{r}_1'(t) = e^t \bar{i} + \bar{j} + \bar{k} \quad \text{and} \quad \bar{n} = \text{angle of intersection} \]

\[ \bar{r}_1'(0) = \bar{i} + \bar{j} + \bar{k} \]

\[ \bar{r}_2'(t) = \bar{i} + \bar{j} - 2t \bar{k} \]

\[ \bar{r}_2'(1) = \bar{i} + \bar{j} - 2 \bar{k} \]

\[ \cos \theta = \frac{\bar{r}_1'(0) \cdot \bar{r}_2'(1)}{|\bar{r}_1'(0)||\bar{r}_2'(1)|} \]

\[ \theta = 0 \]

\[ \Rightarrow \text{ angle of intersection is } \frac{\pi}{2} \text{ radians or } 90^\circ. \]
5. (12 points) Find the point(s) on the cone \( z^2 = x^2 + y^2 \) that are closest to the point \((3, 1, 0)\).

\[
\text{constrained min: } S = \text{distance squared to } (3, 1, 0) \\
S = (x-3)^2 + (y-1)^2 + z^2
\]

\[
\text{constraint: } C(x, y, z) = x^2 + y^2 - z^2 = 0.
\]

\[
\nabla S = 2(x-3)\mathbf{i} + 2(y-1)\mathbf{j} + 2z\mathbf{k}
\]

\[
\nabla C = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k}
\]

\[
\nabla S = \lambda \nabla C \Rightarrow z(x-3) = \lambda (2x) \\
(\text{and similarly for } y \text{ and } z)
\]

Note that the point \((0,0,0)\) on cone is special because \(\nabla C(0,0,0) = 0\).

Away from that point, \(z \neq 0\) and the third equation implies \(\lambda = -1\).

\[
\lambda = -1 \Rightarrow x-3 = -\lambda \Rightarrow x = \frac{3}{2}
\]

\[
y-1 = -\lambda \Rightarrow y = \frac{1}{2}
\]

\[
z^2 = x^2 + y^2 = \frac{9}{4} + \frac{1}{4} = \frac{5}{2} \Rightarrow z = \pm \sqrt{\frac{5}{2}}
\]

The two points \((\frac{3}{2}, \frac{1}{2}, \pm \sqrt{\frac{5}{2}})\) are \(\sqrt{5}\) away from \((3, 1, 0)\). They are definitely closer than \((0,0,0)\), which is \(\sqrt{10}\) away from \((3,1,0)\).
6. (12 points) Find the volume of the solid region $R$ lying inside the sphere $x^2 + y^2 + z^2 = 9$ and outside the cone $z^2 = x^2 + y^2$. To be precise,

$$R = \{(x, y, z) | x^2 + y^2 + z^2 \leq 9 \text{ and } z^2 \leq x^2 + y^2\}.$$

**Volume = $\iiint_R 1 \, dV$.**

Use spherical coordinates:

$$\int_0^{2\pi} \int_{\pi/4}^{3\pi/4} \int_0^3 \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta =$$

$$\int_0^{2\pi} \int_0^{3\pi/4} \rho^2 \left[-\cos \phi\right]_{\phi = 0}^{\phi = \pi/4} \, d\rho \, d\theta =$$

$$\int_0^{2\pi} \int_0^{3\pi/4} \rho^2 \left(-1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) \, d\rho \, d\theta =$$

$$\sqrt{2} \int_0^{2\pi} \int_0^{3\pi/4} \rho^2 \, d\rho \, d\theta = \sqrt{2} \int_0^{2\pi} \left[\frac{\rho^3}{3}\right]_{\rho = 0}^{\rho = 3} \, d\theta =$$

$$= 9 \sqrt{2} \int_0^{2\pi} \, d\theta = (2\pi)(9 \sqrt{2}) = 18\sqrt{2} \pi$$
7. (12 points) Note that there is a second part to this problem on the next page.

(a) Fix a radius $R$. Using the surface area formula we discussed in this course, derive the surface area of a hemisphere of the form

$$x^2 + y^2 + z^2 = R^2 \quad \text{with} \quad z \geq 0.$$

The hemisphere is graph $z = g(x,y) = \sqrt{R^2 - x^2 - y^2}$

$$\frac{\partial g}{\partial x} = \frac{x}{\sqrt{R^2 - x^2 - y^2}} (-2x) = \frac{-x}{\sqrt{R^2 - x^2 - y^2}}$$

$$\frac{\partial g}{\partial y} = \frac{-y}{\sqrt{R^2 - x^2 - y^2}}$$

$$dS = \sqrt{\frac{x^2}{R^2 - x^2 - y^2} + \frac{y^2}{R^2 - x^2 - y^2} + 1} \, dA = \frac{R}{\sqrt{R^2 - x^2 - y^2}} \, dA$$

Surface area:

$$\text{surface area} = \iint_D \frac{R}{\sqrt{R^2 - x^2 - y^2}} \, dA$$

Use polar:

$$\iint_D \frac{R}{\sqrt{R^2 - r^2}} \, rdr \, d\theta$$

$$= \int_0^{2\pi} \int_0^R \frac{R}{\sqrt{R^2 - r^2}} \, rdr \, d\theta$$

$$= R \int_0^{2\pi} \left[ \sqrt{u} \right]_{u=0}^{u=R^2} \, d\theta = R \int_0^{2\pi} \, d\theta$$

$$= R^2 (2\pi) = 2\pi R^2.$$
Problem 7 (continued):

(b) With the aid of your result in part (a), calculate the $x$-coordinate $\bar{x}$ of the center of mass of that portion of the sphere $x^2 + y^2 + z^2 = R^2$ that lies in the first octant ($x$, $y$, and $z$ all positive). You can use the integral

$$\int \frac{u^2}{\sqrt{a^2 - u^2}} \, du = -\frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \sin^{-1} \frac{u}{a} + C$$

without justification. Also, recall that $\bar{x}$ is the surface integral of $x$ over the surface divided by the surface area of the surface.

\[ \iint_{S_{\bar{x}}} x \, dS = \iint_{\Omega} x \, \frac{R}{\sqrt{R^2 - x^2 - y^2}} \, dA \quad R \times y \text{-plane} \]

\[ = \int_0^{\pi/2} \int_0^R (r \cos \theta) \frac{R}{\sqrt{R^2 - r^2}} \, r \, dr \, d\theta \]

\[ = \int_0^{\pi/2} R \cos \theta \left( \int_0^R \frac{r^2}{\sqrt{R^2 - r^2}} \, dr \right) \, d\theta \]

Using the integral formula above

\[ \int_0^R \frac{r^2}{\sqrt{R^2 - r^2}} \, dr = \left[ -\frac{r}{2} \sqrt{R^2 - r^2} + \frac{R^2}{2} \sin^{-1} \frac{r}{R} \right]_{r=0}^{r=R} \]

\[ = \left( \frac{R^2}{2} \right) \left( \frac{\pi}{2} \right) \Rightarrow \]

\[ \iint_{S_{\bar{x}}} x \, dS = \left( \frac{\pi R^3}{4} \right) \int_0^{\pi/2} \cos \theta \, d\theta = \frac{\pi R^3}{4} \]

\[ \bar{x} = \left( \frac{\pi R^3}{4} \right) \frac{Z}{\pi R^2} = \frac{R}{8} \]
8. (16 points) Note: This problem has multiple parts on pages 9–12

(a) Compute and classify the critical points of the function $f_1(x, y) = x^2 - 2x + y^2 - 2y$.

\[ \frac{\partial f_1}{\partial x} = 2x - 2 = 0 \Rightarrow x = 1 \quad \text{one critical point} \]
\[ \frac{\partial f_1}{\partial y} = 2y - 2 = 0 \Rightarrow y = 1 \]
\[ H(x, y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow H(1,1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow D = 4 \frac{\partial^2 f}{\partial x^2} > 0 \]
local min

(b) Compute and classify the critical points of the function $f_2(x, y) = 3x - x^3 + y^2$.

\[ \frac{\partial f_2}{\partial x} = 3 - 3x^2 = 0 \Rightarrow 1 - x^2 = 0 \Rightarrow x = \pm 1 \]
\[ \frac{\partial f_2}{\partial y} = 2y = 0 \Rightarrow y = 0 \quad \text{two critical points} \quad (\pm 1, 0) \]
\[ H(x, y) = \begin{pmatrix} -6x & 0 \\ 0 & 2 \end{pmatrix} \]
\[ H(1,0) = \begin{pmatrix} -6 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow D = -12 \Rightarrow \text{saddle} \]
\[ H(-1,0) = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow D = 12 \Rightarrow \text{local min} \]
\[ \frac{\partial^2 f}{\partial x^2} = 0 \]
Problem 8 (continued):

(c) Compute and classify the critical points of the function $f_3(x, y) = x^3 - 3x + y^2$.

\[
\frac{df_3}{dx} = 3x^2 - 3 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1
\]
\[
\frac{df_3}{dy} = 2y = 0 \Rightarrow y = 0 \\
\text{two critical points} \ (\pm 1, 0)
\]

\[
H(x, y) = \begin{pmatrix} 6x & 0 \\ 0 & 2 \end{pmatrix}
\]

$H(1, 0) = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow D = 12$ and $\frac{df_3}{dx} > 0 \Rightarrow \text{local min}$

$H(-1, 0) = \begin{pmatrix} -6 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow D = -12 \Rightarrow \text{saddle}$

(d) Compute and classify the critical points of the function $f_4(x, y) = xy - x - y$.

\[
\frac{df_4}{dx} = y - 1 = 0 \Rightarrow y = 1 \ 	ext{one critical point} \ (1, 1)
\]
\[
\frac{df_4}{dy} = x - 1 = 0 \Rightarrow x = 1
\]

\[
H(x, y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow D = -1 \Rightarrow \text{saddle}
\]
Problem 8 (continued):

(e) Compute and classify the critical points of the function \( f_5(x, y) = x^4 + y^4 - 4xy \).

\[
\frac{\partial f_5}{\partial x} = 4x^3 - 4y = 0 \Rightarrow x^3 - y = 0 \Rightarrow y = x^3 \\
\frac{\partial f_5}{\partial y} = 4y^3 - 4x = 0 \Rightarrow y^3 - x = 0 \Rightarrow x = y^3 \\
x = y^3 \Rightarrow x = (x^3)^3 = x^9 \Rightarrow x^9 - x = 0. \\
x^9 - x = x(x^8 - 1) = x(x^4 - 1)(x^4 + 1) = x(x^2 - 1)(x^2 + 1)(x^4 + 1) = x(x - 1)(x + 1)(x^2 + 1)(x^4 + 1) \\
\Rightarrow x = 0, \pm 1
\]

Three critical points \((1, 1), (0, 0), (-1, -1)\).

\[
H(x, y) = \begin{pmatrix} 12x^2 & 4 \\ 4 & 12y^2 \end{pmatrix}
\]

\(D(0, 0) = -16, \quad \text{saddle} \quad \text{local min} \)

(f) Compute and classify the critical points of the function \( f_6(x, y) = x^4 + y^4 + 4xy \).

Almost same as (e).

\[
\frac{\partial f_6}{\partial x} = 4x^3 + 4y = 0 \Rightarrow y = -x^3 \\
\frac{\partial f_6}{\partial y} = 4y^3 + 4x = 0 \Rightarrow x = -y^3 \Rightarrow x = -(x^3)^3 = x^9 \\
x^9 - x = 0 \leftrightarrow x(x^8 - 1) = 0 \leftrightarrow x(x^4 - 1)(x^4 + 1) = 0 \\
\leftrightarrow x(x^2 - 1)(x^2 + 1)(x^4 + 1) = 0 \\
\leftrightarrow x = 0, \pm 1
\]

Three critical points \((1, 1), (0, 0), (-1, -1)\).

\[
H(x, y) = \begin{pmatrix} 12x^2 & 4 \\ 4 & 12y^2 \end{pmatrix}
\]

\(H(1, 1) = H(-1, -1) = \begin{pmatrix} 12 & 4 \\ 4 & 12 \end{pmatrix} \)

\(D = 144 - 16 > 0, \quad \text{local min} \)
Problem 8 (continued):
(g) The six functions considered in parts (a)-(f) are:

1. \( f_1(x, y) = x^2 - 2x + y^2 - 2y \)
2. \( f_2(x, y) = 3x - x^3 + y^2 \)
3. \( f_3(x, y) = x^3 - 3x + y^2 \)
4. \( f_4(x, y) = xy - x - y \)
5. \( f_5(x, y) = x^4 + y^4 - 4xy \)
6. \( f_6(x, y) = x^4 + y^4 + 4xy \)

Here are three contour plots:

Match each contour plot with its corresponding function \( f(x, y) \) from the choices above. Using your results from parts (a)-(f), provide a brief justification for your selection. You will not receive any credit for your answer unless you provide a valid justification.

A. The function for contour plot A is \( f_6 \). My reason for choosing this answer is:

Three critical points: \((-1,1), (0,0), (1,-1)\).

The only function with three critical points at those places is \( f_6 \).

B. The function for contour plot B is \( f_4 \). My reason for choosing this answer is:

One critical point at \((1,1)\), and it is a saddle. The only function with one critical point that is a saddle is \( f_4 \).

C. The function for contour plot C is \( f_3 \). My reason for choosing this answer is:

Two critical points: \((-1,0)\) and \((1,0)\). \( f_2 \) and \( f_3 \) satisfy this situation. \( f_3 \) has a saddle at \((-1,0)\) and \( f_2 \) does not.