

Evaluating double integrals

At the end of last class, we discussed how we can approximate the value of a double integral using Riemann sums. Today we discuss a process called iterated integration that we can use to calculate many double integrals exactly.

However, before we get started with iterated integration, I want to mention some of the basic properties of the double integral.

Theorem. The integral satisfies some basic but important properties.

1. $\iint_R f(x, y) + g(x, y) \, dA = \iint_R f(x, y) \, dA + \iint_R g(x, y) \, dA$
2. $\iint_R c f(x, y) \, dA = c \iint_R f(x, y) \, dA$ if c is a constant
3. If $f(x, y) \geq g(x, y)$ for all (x, y) in R , then

$$\iint_R f(x, y) \, dA \geq \iint_R g(x, y) \, dA.$$

Iterated integration

Example. Consider

$$\iint_R \frac{y}{3} \, dA$$

where $R = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 3\}$.

We can generalize this slicing technique to obtain a method for calculating double integrals. Consider a positive function $f(x, y)$ and a rectangle

$$R = \{(x, y) \mid a \leq x \leq b, \ c \leq y \leq d\}$$

in the xy -plane. We can calculate the volume of the solid determined by R and $f(x, y)$ using x -slices or y -slices.

Pictures of x -slices and y -slices are available on the course web site and in your textbook as Figures 1 and 2 on p. 839.

The basic idea is to compute the volume of the solid in question by integrating the areas of the slices. For example, suppose that we slice up the solid using y -slices. Let $A(y)$ denote the area of a y -slice. Then

$$\iint_R f(x, y) \, dA = \int_c^d A(y) \, dy.$$

Moreover, for any given y , the area of the y -slice is

$$A(y) = \int_a^b f(x, y) \, dx.$$

We obtain an *iterated integral* that yields the volume of the solid. That is,

$$\iint_R f(x, y) \, dA = \int_c^d \left[\int_a^b f(x, y) \, dx \right] dy.$$

We compute the inside integral treating y as a constant (“partial integration”), and then we compute the outside integral which only depends on y .

Of course, it also possible to use x -slices rather than y -slices.

Theorem. (Fubini’s Theorem) If $f(x, y)$ is continuous on the rectangle

$$R = \{(x, y) \mid a \leq x \leq b, \ c \leq y \leq d\},$$

then

$$\begin{aligned} \iint_R f(x, y) \, dA &= \int_a^b \int_c^d f(x, y) \, dy \, dx \\ &= \int_c^d \int_a^b f(x, y) \, dx \, dy. \end{aligned}$$

Let's return to the example discussed earlier.

Example. Consider

$$\iint_R \frac{y}{3} dA$$

where $R = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 3\}$.

The calculation that we did involved x -slices.

We can also calculate the integral using y -slices.

Example. Consider the double integral

$$\iint_R x \cos(xy) \, dA$$

where $R = \{(x, y) \mid 0 \leq x \leq \pi/4, 0 \leq y \leq 2\}$.

Example. Calculate the average value of the function

$$f(x, y) = xe^{xy}$$

over the rectangle $R = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 2\}$.