Functions of many variables

Many things are best modeled by functions of more than one variable.

Example. The cost of producing an item is a function of the wage rate, the number of hours it takes to produce the item, and the cost of materials.

Visualizing functions of more than one variable

For a function of one variable $y = f(x)$, we tend to visualize it by drawing its graph in the xy-plane. For a function of more than one variable, there are other ways to visualize it.

Example. Consider the function $h(x, y) = 4x^2 + y^2$. What kind of surface is its graph?

Definition. Given a function $f(x, y)$ of two variables, its level set of level K is the set of all points (x, y) such that

$$f(x, y) = K.$$

We see functions that are displayed in terms of their level sets all of the time. A typical example is Figure 6 on p. 742 of your textbook. It displays sea-level temperatures in January. We are also used to visualizing altitude in topographic maps.

Example. For the function $h(x, y) = 4x^2 + y^2$ above, what can we say about its level sets?

Computers are especially good at drawing contour maps.

Example. Consider the function

$$f(x, y) = \frac{-xy}{e^{x^2+y^2}}.$$

Functions of three variables

For a function $f(x, y, z)$ of three variables, its graph would be the graph of

$$w = f(x, y, z).$$

Example. Sketch the level sets of the function

$$P(x, y, z) = x + y + 10z.$$

Example. Sketch the level sets of the function

$$f(x, y, z) = x^2 + y^2 - z^2.$$

Limits and continuity

In order to be able to do calculus for multivariable functions, we need to be able to talk about limits.

Informal definition. We say that

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if $f(x,y) \to L$ as $(x,y) \to (a,b)$ along any path in the xy-plane.

Here are two examples to illustrate some of the issues that arise.

Example. Consider

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}.$$

Example. Consider

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}.$$