Curl and divergence of vector fields

Today we introduce two quantities associated to a vector field. The curl of a vector field measures how much it "spirals." The divergence of a vector field measures how fast it "expands."

Definition. Given a vector field in space

$$\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k},$$

the curl of $\mathbf{F}$ is another vector field defined by

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

This formula seems quite nasty, but there is an easy way to remember it.

$$\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x, y, z) & Q(x, y, z) & R(x, y, z) \end{vmatrix}.$$

Example. Calculate $\operatorname{curl}\mathbf{F}$ for $\mathbf{F}(x, y, z) = x^2y\mathbf{i} + yz^2\mathbf{j} + x^2z\mathbf{k}$.

The formula for the curl is related to the question of whether a vector field in space has a potential function.

Theorem. If the vector field $\mathbf{F}(x, y, z)$ in space is the gradient of some function $f(x, y, z)$, then

$$\operatorname{curl} \mathbf{F} = \mathbf{0}.$$

Sometimes you will see this fact expressed very succinctly as

$$\nabla \times \nabla f = \mathbf{0}.$$

The theorem gives a necessary condition for a vector field in space to have a potential function. As in the case of vector fields in the plane, this necessary condition is also sufficient if the vector field is continuously differentiable everywhere.

Theorem. Suppose the vector field $\mathbf{F}$ is defined and continuously differentiable for all (x, y, z). If

$$\operatorname{curl} \mathbf{F} = \mathbf{0}$$

for all (x, y, z), then $\mathbf{F}$ has a potential function.

The potential function is calculated using a generalization of the procedure that we discussed last class.

There is an animation of the concept of curl from the University of Minnesota that is referenced on the class web site.

I would like to use Green's Theorem to explain the concept of curl if $\mathbf{F}$ is essentially a planar field. To do so, I would like you to consider the vector field $\mathbf{F}$ as a velocity field of a fluid. In this case, it is helpful to imagine the flowlines or streamlines of the fluid.

Given a curve $\mathbf{r}(t)$, the line integral

$$\int_{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{r}} \mathbf{F} \cdot \mathbf{T}\, ds.$$

For velocity fields of fluids, this line integral is called the *circulation* of the fluid along the curve.

Here are pictures of three examples.

To interpret the curl of $\mathbf{F}$ in this situation, we use Green's Theorem.

Theorem. (The vector form of Green's Theorem) Let C be a positively-oriented, simple, closed curve in the xy-plane and let D be the region that is enclosed by C. Then

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_D (\operatorname{curl}\mathbf{F})\cdot\mathbf{k}\,dA.$$

Divergence

Definition. Given a vector field in space

$$\mathbf{F}(x,y,z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k},$$

the divergence of $\mathbf{F}$ is the scalar field (scalar function) defined by

$$\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Shorthand notation: $\operatorname{div}\mathbf{F} = \nabla\boldsymbol{\cdot}\mathbf{F}$.

Example. Calculate $\operatorname{div}\mathbf{F}$ for $\mathbf{F}(x,y,z) = x^2y\mathbf{i} + yz^2\mathbf{j} + x^2z\mathbf{k}$.

To understand what divergence measures in the case of our velocity field of a fluid, we consider a different path integral. Given a simple, closed curve C in the plane, consider the path integral

$$\oint_C \mathbf{F}\boldsymbol{\cdot}\mathbf{n}\,ds,$$

where $\mathbf{n}$ is a unit normal vector to C that points outside the region enclosed by C.

Theorem. (planar Divergence Theorem) Let C be a positively-oriented, simple, closed curve in the xy-plane and let D be the region that is enclosed by C. Then

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \, dA = \iint_D (\operatorname{div} \mathbf{F}) \, dA.$$

This identity justifies the name "divergence."