A little more about arc length and the derivative

Last class we discussed the arc length formula

$$\text{arc length} = \int_a^b |\mathbf{f}'(t)|\, dt.$$

There is an important interpretation of $|\mathbf{f}'(t)|$ that follows from this formula.

If the vector-valued function $\mathbf{f}(t)$ represents motion in space such as the path of an airplane, then the arc length would be the distance travelled, and its derivative would be its speed. If we differentiate our formula for arc length, we are able to conclude that

$$|\mathbf{f}'(t)| = \text{speed}.$$

Consequently, the limit

$$\mathbf{f}'(t) = \lim_{h \to 0} \frac{\mathbf{f}(t+h) - \mathbf{f}(t)}{h}$$

produces a vector that has two nice properties:

1. Its direction is tangent to the curve.
2. Its length is the speed of the motion.

On the web site, there is an animation involving the trefoil knot curve where the point speeds up and slows down. If you look carefully, you can see that the length of the derivative changes as well.

Review of curves defined using polar coordinates

Associated to any pair of numbers (r, θ) is a point in the plane determined as follows:

With polar coordinates, we allow negative r, which sometimes produces unexpected results (as we shall see).

Basic graphs:

Example 1. $r = 3$

Example 2. $\theta = \pi/4$

Let's do two more examples that are a little more complicated.

Example 3. $r = 3\cos\theta$

Example 4. $r = \cos 2\theta$

In this course, polar coordinates are used to describe curves, regions, and functions that have a lot of radial symmetry, and we need to be able to convert from rectangular coordinates to polar coordinates and vice versa. The conversion formulas all come from trigonometry.

Example 3 revisited. How can we convert $r = 3\cos\theta$ to rectangular coordinates?

Example 5. $r = 1 + 2\cos\theta$