Vectors

A *vector* is an arrow—it has direction and length. It is used to measure quantities such as displacements, forces, and velocities. For example, if you are hiking and you say that you are 3 miles northwest of your camp, you are specifying a vector.

The precise mathematical definition of a vector is:

A *vector* is a directed line segment.

The length of a vector $\mathbf{V}$ is sometimes called its magnitude or the norm of $\mathbf{V}$, and we shall always abbreviate this quantity by the symbol $|\mathbf{V}|$.

Definition. Two vectors are *equal* if they point in the same direction and if they have the same length.

Note that the locations of the initial points of the vectors are irrelevant (unless we are talking solely about position vectors—see below).

Vector addition. We can add vectors:

Scalar Multiplication. We can multiply vectors by real numbers:

If $\alpha > 0$, then $\alpha\mathbf{A}$ is the vector in the direction of $\mathbf{A}$ whose length is $\alpha|\mathbf{A}|$. If $\alpha < 0$, then the vector $\alpha\mathbf{A}$ satisfies the equations direction $(\alpha\mathbf{A}) = -$direction $(\mathbf{A})$ and $|\alpha\mathbf{A}| = (-\alpha)|\mathbf{A}|$.

Since we can add vectors and perform scalar multiplication, we can subtract two vectors:

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-1)\mathbf{B}$$

We can derive the same vector by using the equation

$$\mathbf{B} + (\mathbf{A} - \mathbf{B}) = \mathbf{A}.$$

Example. Suppose that the wind is blowing due west at 40 miles/hour. In what direction should a pilot steer an airplane flying with an air speed of 200 miles/hour in order to maintain a course that is due north?

Up to this point, our entire emphasis has been geometric. Now, it is worthwhile to introduce coordinates for vectors and to reinterpret the geometric operations defined above in terms of coordinates.

Definition. A *position vector* is a vector that starts at the origin.

There is a one-to-one correspondence between points in the plane and position vectors in the plane as well as for position vectors and points in space.

We can also view this picture in terms of the unit vectors **i**, **j**, and **k**.

Then $\mathbf{A} = a_i\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$.

Given a vector $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, we can use the distance formula to derive

$$|\mathbf{A}| = \sqrt{(a_1 - 0)^2 + (a_2 - 0)^2 + (a_3 - 0)^2}$$

$$= \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

If $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then

$$\mathbf{A} + \mathbf{B} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) + (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$$

$$= (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k}.$$

Moreover,

$$\alpha\mathbf{A} = \alpha a_1\mathbf{i} + \alpha a_2\mathbf{j} + \alpha a_3\mathbf{k}.$$

Equations for lines (using vectors)

Given a line in the plane determined by the two points $P_1 = (a_1, b_1)$ and $P_2 = (a_2, b_2)$, we usually find its equation by first calculating the slope.

$$\text{slope} = m = \frac{\Delta y}{\Delta x} = \frac{b_2 - b_1}{a_2 - a_1}$$

Once we calculate m, we can find the equation of the line using the fact that

$$\frac{y - b_1}{x - a_1} = m$$

for all points (x, y) on the line.

Question: When do we have trouble with this method?

Using vectors, we can describe *all lines*, and the same method works in both the plane and three-dimensional space. We use the two points to form a *direction vector* **D**. Then we can get an equation for the line in terms of position vectors.

The position vector **P** can be expressed uniquely as

$$\mathbf{P}_1 + t\mathbf{D}$$

for the appropriate choice of the scalar t. Any line **L** (thought of as a continuous series of position vectors) can be written as $\mathbf{L}(t) = \mathbf{P}_1 + t\mathbf{D}$ where $\mathbf{D} = \mathbf{P}_2 - \mathbf{P}_1$.

This is the approach that generalizes to three dimensions (or to an arbitrary number of dimensions) most readily. We start the three-dimensional discussion with the assumption that two points in space determine a line.

Any point on the line can be represented as a vector sum. Let

$$\mathbf{P}_1 = (a_1, a_2, a_3) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

and **D** be the vector between the points A and B. Then

$$\mathbf{D} = (b_1 - a_1)\mathbf{i} + (b_2 - a_2)\mathbf{j} + (b_3 - a_3)\mathbf{k}.$$

Given any point P on the line there is a scalar t such that

$$\mathbf{P} = \mathbf{P}_1 + t\mathbf{D}.$$

Consequently, we have a vector equation for the line

$$\mathbf{L}(t) = \mathbf{P}_1 + t\mathbf{D}.$$

The vector **D** is called a *direction vector* for the line.

Example. Find a vector equation for the line containing $(1, 1, 2)$ and $(3, 7, -2)$.