1. (21 points) Calculate:

(a) The angle of intersection of the two lines

\[ L_1(t) = (1+t)i + (1+t)j + (1+\sqrt{6}t)k \quad \text{and} \quad L_2(t) = (1-t)i + (1-t)j + (1+\sqrt{6}t)k. \]

\[
\overrightarrow{D_1} = \overrightarrow{L_1(1)} = i + j + \sqrt{6}k \\
\overrightarrow{D_2} = \overrightarrow{L_2(1)} = -i - j + \sqrt{6}k \\
\cos \theta = \frac{\overrightarrow{D_1} \cdot \overrightarrow{D_2}}{|\overrightarrow{D_1}| |\overrightarrow{D_2}|} = \frac{-1 \cdot -1 + 6}{\sqrt{6} \cdot \sqrt{6}} = \frac{7}{6} = \frac{7}{6} \\
\Rightarrow \theta = \frac{3\pi}{2} \text{ rad}.
\]

(b) Calculate a vector tangent to the trace of \( r(t) = \cos t\ i + t^2\ j + \sin t\ k \) at the point \((0, \pi^2/4, -1)\).

\((0, \pi^2/4, -1) \Rightarrow t = \pm \pi/2 \Rightarrow t = -\pi/2\)

\[ r'(t) = (-\sin t)\ i + (2t)\ j + (\cos t)\ k \]

\[ \text{tangent vector} \quad r'(-\pi/2) = i - \pi/2 j. \]

(c) All unit vectors that are orthogonal to the plane \( x - 2y + 4z = 3 \).

\[ \text{normal vector} \quad \overrightarrow{n} = i - 2j + 4k \]

\[ \text{unit normal} \quad \hat{n} = \frac{1}{\sqrt{21}} (i - 2j + 4k) \]
2. (18 points) Calculate the arc length of the curve swept out by the vector-valued function
\[ \mathbf{r}(t) = t^2 \mathbf{i} + 2t \mathbf{j} + \ln t \mathbf{k} \]
from the point \((1, 2, 0)\) to the point \((e^2, 2e, 1)\).

\[ t = 1 \quad t = e \]

\[ \mathbf{r}'(t) = (2t) \mathbf{i} + 2 \mathbf{j} + \left( \frac{1}{t} \right) \mathbf{k} \]

\[ |\mathbf{r}'(t)| = \sqrt{4t^2 + 4 + \frac{1}{t^2}} \]

\[ = \sqrt{(2t + \frac{1}{t})^2} = 2t + \frac{1}{t} \quad \text{for} \quad 1 \leq t \leq e \]

\[ \text{Arc length} = \int_{1}^{e} |\mathbf{r}'(t)| \, dt \]

\[ = \int_{1}^{e} (2t + \frac{1}{t}) \, dt \]

\[ = \left[ t^2 + \ln t \right]_{t=1}^{t=e} \]

\[ = (e^2 + 1) - (1 + 0) = e^2 \]
3. (18 points) Find an equation for the plane that contains the point \((1, -1, 1)\) and the line \(x = 2y = 3z\).

Direction vector for line: \(\frac{1}{6} = \frac{y}{3} = \frac{z}{2}\)

\[\vec{D} = 6\hat{i} + 3\hat{j} + 2\hat{k}\]

Point \((0, 0, 0)\) on the line, so another vector in the plane is \(\vec{P} = \hat{i} - \hat{j} + \hat{k}\)

Normal vector for plane

\[\vec{N} = \vec{D} \times \vec{P} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
6 & 3 & 2 \\
1 & -1 & 1
\end{vmatrix}
\]

\[= 5\hat{i} - 4\hat{j} - 9\hat{k}\]

Let \(Q = (x, y, z)\) be the plane and use point \(P = (0, 0, 0)\).

Then \(\vec{N} \cdot \vec{PQ} = 0 \Rightarrow 5x - 4y - 9z = 0\).
4. (20 points) Find the area of the region that is inside one loop of the polar curve 
\( r = 2 \sin 2\theta \) but outside the circle \( r = 1 \). (The half-angle formula for sine is \( \sin^2 \theta = \frac{1}{2} (1 - \cos 2\theta) \).)

\[ \theta = 0 \Rightarrow r = 0 \]
Also \( r = 0 \iff \sin 2\theta = 0 \)
\[ \iff 2\theta = \pi, 2\pi, \ldots \]

One loop is \( 0 \leq \theta \leq \pi/2 \)

Want \( \theta \) such that

\[ 2 \sin 2\theta = 1 \] in first quadrant.

\[ \Rightarrow \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6} \text{ or } \frac{\pi}{6} \]

\[ \Rightarrow \theta = \frac{\pi}{12} \text{ or } \frac{5\pi}{12} \]

Area = \[ \frac{1}{2} \int_{\pi/12}^{5\pi/12} (2 \sin 2\theta)^2 - 1^2 \, d\theta \]

\[ \frac{1}{2} \int_{\pi/12}^{5\pi/12} 4 \sin^2 2\theta \, d\theta = \frac{1}{2} \int_{\pi/12}^{5\pi/12} 2(1 - \cos 4\theta) \, d\theta \]

\[ \int_{\pi/12}^{5\pi/12} (1 - \cos 4\theta) \, d\theta = \int_{\pi/3}^{5\pi/3} (1 - \cos u)(\frac{1}{4}) \, du \]

\[ u = 4\theta \]
\[ du = 4\, d\theta \]
\[
\frac{1}{2} \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} 4 \sin^2 2\theta \, d\theta = \frac{1}{4} \left[ u - \sin u \right]_{u = \frac{\pi}{3}}^{\frac{\pi}{2}} \\
= \frac{1}{4} \left[ \left( \frac{5\pi}{3} + \frac{\sqrt{3}}{2} \right) - \left( \frac{\pi}{3} - \frac{\sqrt{3}}{2} \right) \right] \\
= \frac{1}{4} \left[ \frac{4\pi}{3} + \frac{\sqrt{3}}{2} \right] \\
= \frac{\pi}{3} + \frac{\sqrt{3}}{4} \\

\frac{1}{2} \int_{\frac{\pi}{12}}^{\frac{\pi}{2}} 1 \, d\theta = \left( \frac{1}{2} \right) \left( \frac{4\pi}{12} \right) = \frac{\pi}{6}
\]

\text{desired area} = \left( \frac{\pi}{3} + \frac{\sqrt{3}}{4} \right) - \frac{\pi}{6} \\
= \frac{\pi}{6} + \frac{\sqrt{3}}{4}.
5. (20 points) Here are two surfaces in space:

Here are six possible parametrizations:

1. \( r_1(u, v) = u \mathbf{i} + u \cos v \mathbf{j} + u \sin v \mathbf{k} \)
2. \( r_2(u, v) = \sin u \cos v \mathbf{i} + \sin u \sin v \mathbf{j} + \cos u \mathbf{k} \)
3. \( r_3(u, v) = u \cos v \mathbf{i} + u \mathbf{j} + u \sin v \mathbf{k} \)
4. \( r_4(u, v) = u \mathbf{i} + \sqrt{1 + u^2} \cos v \mathbf{j} + \sqrt{1 + u^2} \sin v \mathbf{k} \)
5. \( r_5(u, v) = u^2 \cos v \mathbf{i} + u \mathbf{j} + u^2 \sin v \mathbf{k} \)
6. \( r_6(u, v) = \sqrt{1 + u^2} \cos v \mathbf{i} + u \mathbf{j} + \sqrt{1 + u^2} \sin v \mathbf{k} \)

For each surface, pick the corresponding parametrization. Provide a brief justification for your choice. You will not receive any credit unless you provide a valid justification.

(a) The parametrization for surface A is 5. My reason for choosing this answer is:

Surface A contains the origin \((0, 0, 0)\) \(\Rightarrow\) #1, #3, #5.
#1 is a surface of revolution about the x-axis. #3 is a circular cone. #5 is the graph of \(z = \frac{1}{u^2}\) revolved about the y-axis.

(b) The parametrization for surface B is 6. My reason for choosing this answer is:

Surface B does not contain \((0, 0, 0)\) \(\Rightarrow\) #2, #4, #6.
#2 is the unit sphere by spherical coordinates. #6 is a surface of revolution about the y-axis. #4 is a circular hyperboloid of one sheet. In this case it is a surface of revolution about the x-axis.