1. (21 points) Calculate:

(a) The angle of intersection of the two lines

$$\mathbf{L}_1(t) = (1+t)\mathbf{i}+(1+t)\mathbf{j}+(1+\sqrt{6}\,t)\mathbf{k} \quad \text{and} \quad \mathbf{L}_2(t) = (1-t)\mathbf{i}+(1-t)\mathbf{j}+(1+\sqrt{6}\,t)\mathbf{k}.$$

$\vec{D_1} = \vec{i} + \vec{j} + \sqrt{6}\,\vec{k}$

$\vec{D_2} = -\vec{i} - \vec{j} + \sqrt{6}\,\vec{k}$

$\cos\theta = \dfrac{\vec{D_1}\cdot\vec{D_2}}{|\vec{D_1}||\vec{D_2}|} = \dfrac{-1-1+6}{\sqrt{8}\sqrt{8}} = \dfrac{4}{8} = \dfrac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{3}$ rad.

(b) Calculate a vector tangent to the trace of $\mathbf{r}(t) = \cos t\,\mathbf{i} + t^2\,\mathbf{j} + \sin t\,\mathbf{k}$ at the point $(0, \pi^2/4, -1)$.

$(0, \frac{\pi^2}{4}, -1) \Rightarrow t = \pm\frac{\pi}{2} \Rightarrow t = -\frac{\pi}{2}$

$\vec{r}'(t) = (-\sin t)\vec{i} + (2t)\vec{j} + (\cos t)\vec{k}$

tangent vector $\vec{r}'(-\frac{\pi}{2}) = \vec{i} - \pi\vec{j}$.

(c) All unit vectors that are orthogonal to the plane $x - 2y + 4z = 3$.

normal vector $\vec{N} = \vec{i} - 2\vec{j} + 4\vec{k}$

unit normal $\pm\dfrac{\vec{N}}{|\vec{N}|} = \pm\dfrac{1}{\sqrt{21}}(\vec{i} - 2\vec{j} + 4\vec{k})$

2. (18 points) Calculate the arc length of the curve swept out by the vector-valued function

$$\mathbf{r}(t) = t^2\,\mathbf{i} + 2t\,\mathbf{j} + \ln t\,\mathbf{k}$$

from the point $(1, 2, 0)$ to the point $(e^2, 2e, 1)$.

$t=1$ $t=e$

$$\vec{r}\,'(t) = (2t)\vec{i} + 2\vec{j} + \left(\frac{1}{t}\right)\vec{k}$$

$$|\vec{r}\,'(t)| = \sqrt{4t^2 + 4 + \frac{1}{t^2}}$$

$$= \sqrt{\left(2t + \frac{1}{t}\right)^2} = 2t + \frac{1}{t}$$

for $1 \le t \le e$

$$\text{arclength} = \int_1^e |\vec{r}\,'(t)|\,dt$$

$$= \int_1^e \left(2t + \frac{1}{t}\right) dt$$

$$= \Big[t^2 + \ln t\Big]_{t=1}^{t=e}$$

$$= (e^2 + 1) - (1 + 0) = e^2$$

3. (18 points) Find an equation for the plane that contains the point $(1, -1, 1)$ and the line $x = 2y = 3z$.

Direction vector for line: $\frac{x}{6} = \frac{y}{3} = \frac{z}{2}$

$\vec{D} = 6\vec{i} + 3\vec{j} + 2\vec{k}$

Point $(0,0,0)$ on the line, so another vector in the plane is

$\vec{P} = \vec{i} - \vec{j} + \vec{k}$

Normal vector for plane

$$\vec{N} = \vec{D} \times \vec{P} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & 3 & 2 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= 5\vec{i} - 4\vec{j} - 9\vec{k}$$

Let $Q = (x, y, z)$ in the plane and use point $P = (0,0,0)$

Then $\vec{N} \cdot \vec{PQ} = 0 \Rightarrow$

$5x - 4y - 9z = 0.$

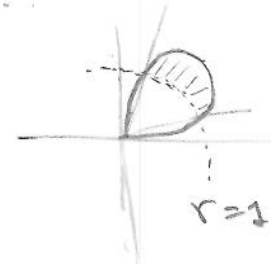
4. (20 points) Find the area of the region that is inside one loop of the polar curve $r = 2\sin 2\theta$ but outside the circle $r = 1$. (The half-angle formula for sine is $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$.)

$\theta = 0 \Rightarrow r = 0$

Also $r = 0 \Leftrightarrow \sin 2\theta = 0$

$\Leftrightarrow 2\theta = \pi, 2\pi, \ldots$

One loop is $0 \le \theta \le \pi/2$



Want θ such that

$2\sin 2\theta = 1$ in first quadrant.

$\Rightarrow \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

$\Rightarrow \theta = \frac{\pi}{12}$ or $\frac{5\pi}{12}$.

$$\text{area} = \frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} (2\sin 2\theta)^2 - 1^2 \, d\theta$$

$$\frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} 4\sin^2 2\theta \, d\theta = \cancel{\frac{1}{2}}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \cancel{2}(1 - \cos 4\theta)\, d\theta$$

$$\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} (1 - \cos 4\theta)\, d\theta = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - \cos u)\left(\frac{1}{4}\right) du$$

$u = 4\theta$

$du = 4\,d\theta$

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$$\frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} 4\sin^2 2\theta \, d\theta = \frac{1}{4}\Big[u - \sin u\Big]_{u=\frac{\pi}{3}}^{u=\frac{5\pi}{3}}$$

$$= \frac{1}{4}\left[\left(\frac{5\pi}{3} + \frac{\sqrt{3}}{2}\right) - \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{1}{4}\left[\frac{4\pi}{3} + \sqrt{3}\right]$$

$$= \frac{\pi}{3} + \frac{\sqrt{3}}{4}$$

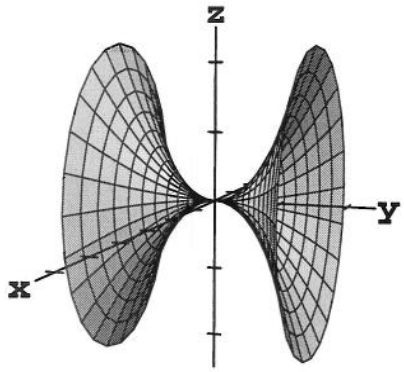
$$\frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} 1 \, d\theta = \left(\frac{1}{2}\right)\left(\frac{4\pi}{12}\right) = \frac{\pi}{6}$$

$$\text{desired area} = \left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right) - \frac{\pi}{6}$$

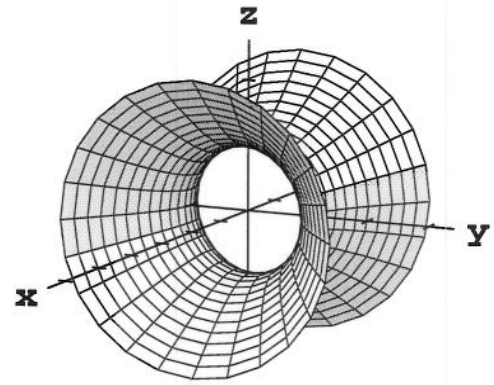
$$= \frac{\pi}{6} + \frac{\sqrt{3}}{4}.$$

5. (20 points) Here are two surfaces in space:

Surface A



Surface B



Here are six possible parametrizations:

1. $\mathbf{r}_1(u,v) = u\,\mathbf{i} + u\cos v\,\mathbf{j} + u\sin v\,\mathbf{k}$
2. $\mathbf{r}_2(u,v) = \sin u\cos v\,\mathbf{i} + \sin u\sin v\,\mathbf{j} + \cos u\,\mathbf{k}$ sphere by spherical coords.
3. $\mathbf{r}_3(u,v) = u\cos v\,\mathbf{i} + u\,\mathbf{j} + u\sin v\,\mathbf{k}$
4. $\mathbf{r}_4(u,v) = u\,\mathbf{i} + \sqrt{1+u^2}\cos v\,\mathbf{j} + \sqrt{1+u^2}\sin v\,\mathbf{k}$
5. $\mathbf{r}_5(u,v) = u^2\cos v\,\mathbf{i} + u\,\mathbf{j} + u^2\sin v\,\mathbf{k}$
6. $\mathbf{r}_6(u,v) = \sqrt{1+u^2}\cos v\,\mathbf{i} + u\,\mathbf{j} + \sqrt{1+u^2}\sin v\,\mathbf{k}$

For each surface, pick the corresponding parametrization. Provide a brief justification for your choice. **You will not receive any credit unless you provide a valid justification.**

(a) The parametrization for surface A is 5. My reason for choosing this answer is:

Surface A contains the origin $(0,0,0) \Rightarrow$ #1, 3, 5. #1 is a surface of revolution about the x-axis. #3 is a circular cone. #5 is the graph of $z = y^2$ revolved about the y-axis.

(b) The parametrization for surface B is 4. My reason for choosing this answer is:

Surface B does not contain $(0,0,0) \Rightarrow$ #2, 4, 6. #2 is the unit sphere by spherical coordinates. #6 is a surface of revolution about the y-axis. #4 is a circular hyperboloid of one sheet. In this case it is a surface of revolution about the x-axis.