1. (19 points) Calculate the average value of the function \( f(x, y) = e^{2x^2+2y^2} \) over the region that is inside the circle \( x^2 + y^2 = 3 \) and above the line \( y = x \).

Converting to polar coordinates,

\[
\iint_R e^{2x^2 + 2y^2} \, dA =
\]

\[
\int_{\pi/4}^{5\pi/4} \int_0^{\sqrt{3}} e^{r^2} \, r \, dr \, d\theta
\]

\[
= \int_{\pi/4}^{5\pi/4} \int_0^6 e^u \left( \frac{1}{4} \right) du \, d\theta
\]

\[
= \int_{\pi/4}^{5\pi/4} \frac{e^6 - 1}{4} \, d\theta = \frac{\pi}{4} (e^6 - 1)
\]

Area of region = \( \left( \frac{1}{2} \cdot \pi \right) (3) = \frac{3}{2} \pi \)

Average value = \[
\frac{\frac{\pi}{4} (e^6 - 1)}{\frac{3}{2} \pi} = \frac{e^6 - 1}{6}
\]
2. (19 points) The table below contains rainfall readings in inches over a 600-foot-by-200-foot rectangular field (600 feet east-west and 200 feet north-south—see a sketch of the field immediately below).

\[ \Delta x = 200 \text{ ft} \]
\[ \Delta y = 100 \text{ ft} \]

The readings were made over a rectangular grid at regular intervals, and they are recorded in the table using a rectangular coordinate system where the origin represents the southwest corner of the region, \( x \) represents the east-west distance of the grid point to the origin, and \( y \) represents the north-south distance of the grid point to the origin.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>100</th>
<th>200</th>
<th>300</th>
<th>400</th>
<th>500</th>
<th>600</th>
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<td>2.2</td>
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<td>2.6</td>
<td>2.8</td>
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<td>2.3</td>
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<td>2.6</td>
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<td>2.8</td>
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<tr>
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<td>2.3</td>
<td>2.5</td>
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<tr>
<td>150</td>
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<td>2.7</td>
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<td>2.8</td>
<td>2.9</td>
</tr>
<tr>
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<td>2.5</td>
<td>2.5</td>
<td>2.5</td>
<td>2.6</td>
<td>2.7</td>
<td>2.9</td>
<td>3.0</td>
</tr>
</tbody>
</table>

Estimate the total amount of rain that fell on the entire field using a Riemann sum with three east-west subdivisions (\( m = 3 \)) and two north-south subdivisions (\( n = 2 \)). As sample/test points, use the points that are the farthest north and east in each subrectangle. Express your answer in cubic feet.

Riemann sum = \((2.6 + 2.8 + 2.7 + 2.7 + 3.0)(\Delta x)(\Delta y)\)

= \((16.3 \text{ inches})(200 \text{ ft})(100 \text{ ft})\)

= 27,166.67 \text{ ft}^3
3. (19 points) The following table gives the heat index $I$ (perceived temperature) as a function of actual temperature $T$ ($90 \leq T \leq 100$) and relative humidity $H$ ($50 \leq H \leq 90$).

<table>
<thead>
<tr>
<th></th>
<th>50</th>
<th>55</th>
<th>60</th>
<th>65</th>
<th>70</th>
<th>75</th>
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<tbody>
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<td>111</td>
<td>114</td>
<td>118</td>
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<td>127</td>
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<td>123</td>
<td>127</td>
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<td>135</td>
<td>141</td>
<td>147</td>
<td>154</td>
<td>161</td>
<td>168</td>
</tr>
</tbody>
</table>

(a) Use the table to find a linear approximation of the heat index $I$ when the temperature is near 96° and the relative humidity is near 80%.

$$\begin{align*}
estimate \quad \frac{\Delta I}{\Delta T} &= \frac{141 - 127}{4} = \frac{14}{4} = 4.25 \\
estimate \quad \frac{\Delta I}{\Delta H} &= \frac{141 - 130}{10} = \frac{11}{10} = 1.1
\end{align*}$$

heat index $\approx 135 + (4.25)(T - 96) + (1.1)(H - 80)$

(b) Estimate the heat index when the temperature is 97° and the relative humidity is 78%.

$$\begin{align*}
T = 97° & \implies \Delta T = 1° \\
H = 78° & \implies \Delta H = -2°
\end{align*}$$

heat index $\approx 135 + (4.25)(1) + (1.1)(-2)$

$T = 97°$
$H = 78°$

$= 137.65$
4. (20 points) Find all critical points of the function \( f(x, y) = x^3 + y^3 + 12xy \) and classify them using the Second Partials Test.

\[
\begin{align*}
\frac{\partial f}{\partial x} &= 3x^2 + 12y = 0 \implies 3x^2 = -12y \\
\frac{\partial f}{\partial y} &= 3y^2 + 12x = 0 \implies 3y^2 = -12x
\end{align*}
\]

\[ x^2 = -4y \quad \text{and} \quad y^2 = -4x \]

\[ \implies x^4 = 16y^2 \implies x^4 = 16 \implies x = \pm 4 \]

\[ \implies x^4 + 64x = 0 \implies x(x^3 + 64) = 0 \]

\[ \implies x = 0 \quad \text{or} \quad x = -4 \cdot \sqrt[3]{64} = -4 \]

\[ \begin{align*}
x = 0 \implies y &= 0 \\
(x, y) &= (0, 0)
\end{align*} \]

\[ \begin{align*}
x = -4 \implies y &= -4 \\
(x, y) &= (-4, -4)
\end{align*} \]

Hessian matrix \( \frac{\partial^2 f}{\partial x^2} \)

\[
\begin{pmatrix}
6x & 12 \\
12 & 6y
\end{pmatrix}
\]

\[ A = \begin{pmatrix} 0 & 12 \\ 12 & 0 \end{pmatrix} \]

\[ D = \det(A) = 144 \quad \text{saddle} \]

\[ A = \begin{pmatrix} -24 & 12 \\ 12 & -24 \end{pmatrix} \]

\[ D = \det(A) = -432 \]

\[ \frac{\partial^2 f}{\partial x^2} < 0 \implies \text{local maximum} \]
5. (20 points) Find the points on the hyperboloid of one sheet

\[ x^2 + 4y^2 - z^2 = 4 \]

where the tangent plane is parallel to the plane \(2x + 2y + z = 5\).

Let \( f(x, y, z) = x^2 + 4y^2 - z^2 \). Surface is level set of level 4.

\[ \nabla f = (2x)i + (8y)j + (-2z)k \]

Plane has normal vector \( \vec{N} = 2i + 2j + k \).

Want points where \( \nabla f = \lambda \vec{N} \) for some scalar \( \lambda \).

\[ \Rightarrow 2x = 2\lambda \quad \Rightarrow x = 4y = -2z. \]

\[ 8y = 2\lambda \]

\[ -2z = \lambda \]

To find the points:

\[ x^2 + 4\left(\frac{x}{4}\right)^2 - \left(-\frac{x}{2}\right)^2 = 4 \]

\[ x^2 + \frac{x^2}{4} - \frac{x^2}{4} = 4 \quad \Rightarrow x = \pm 2. \]

Two points:

\( (2, \frac{1}{2}, -1) \) and \( (-2, -\frac{1}{2}, 1) \).