1. (19 points) The table below contains rainfall readings in inches over a 600-foot-by-200-foot rectangular field (600 feet east-west and 200 feet north-south—see a sketch of the field immediately below).

![Sketch of rectangular field](image)

The readings were made over a rectangular grid at regular intervals, and they are recorded in the table using a rectangular coordinate system where the origin represents the southwest corner of the region, $x$ represents the east-west distance of the grid point to the origin, and $y$ represents the north-south distance of the grid point to the origin.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>100</th>
<th>200</th>
<th>300</th>
<th>400</th>
<th>500</th>
<th>600</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.1</td>
<td>3.2</td>
<td>3.2</td>
<td>3.3</td>
<td>3.5</td>
<td>3.6</td>
<td>3.8</td>
</tr>
<tr>
<td>50</td>
<td>3.2</td>
<td>3.3</td>
<td>3.4</td>
<td>3.5</td>
<td>3.6</td>
<td>3.9</td>
<td>3.8</td>
</tr>
<tr>
<td>100</td>
<td>3.3</td>
<td>3.5</td>
<td>3.6</td>
<td>3.7</td>
<td>3.8</td>
<td>3.9</td>
<td>3.7</td>
</tr>
<tr>
<td>150</td>
<td>3.4</td>
<td>3.6</td>
<td>3.7</td>
<td>3.8</td>
<td>3.8</td>
<td>3.9</td>
<td>3.9</td>
</tr>
<tr>
<td>200</td>
<td>3.5</td>
<td>3.5</td>
<td>3.5</td>
<td>3.6</td>
<td>3.7</td>
<td>3.9</td>
<td>4.0</td>
</tr>
</tbody>
</table>

Estimate the total amount of rain that fell on the entire field using a Riemann sum with three east-west subdivisions ($m = 3$) and two north-south subdivisions ($n = 2$). As sample/test points, use the points that are the farthest south and east in each subrectangle. Express your answer in cubic feet.

Riemann Sum $= (3.2 + 3.5 + 3.8 + 3.6 + 3.8 + 3.7)(\Delta x)(\Delta y)$

$= (21.6 \text{ inches})(200 \text{ ft})(100 \text{ ft})$

$= 3,600,000 \text{ ft}^3$
2. (19 points) Calculate the average value of the function \( f(x, y) = e^{3x^2 + 3y^2} \) over the region that is inside the circle \( x^2 + y^2 = 5 \) and above the line \( y = x \).

\[
\iint_R e^{3x^2 + 3y^2} \, dA = \int_{\pi/4}^{5\pi/4} \int_0^{\sqrt{5}} e^{3r^2} \, r \, dr \, d\theta
\]

\[
= \int_{\pi/4}^{5\pi/4} \left[ \frac{e^{3r^2}}{6} \right]_0^{\sqrt{5}} \, d\theta
\]

\[
= \int_{\pi/4}^{5\pi/4} \left( \frac{e^{15} - 1}{6} \right) \, d\theta = \frac{\pi}{6} (e^{15} - 1)
\]

\[
\text{area of region} = \frac{1}{2}(\pi)(5) = \frac{5}{2}\pi
\]

\[
\text{average value} = \frac{\frac{\pi}{6} (e^{15} - 1)}{\frac{5\pi}{2}} = \frac{2}{5}(\pi)(e^{15} - 1)
\]

\[
= \frac{e^{15} - 1}{15}
\]
3. (19 points) The following table gives the heat index $I$ (perceived temperature) as a function of actual temperature $T$ ($90 \leq T \leq 100$) and relative humidity $H$ ($50 \leq H \leq 90$).

<table>
<thead>
<tr>
<th>T</th>
<th>50</th>
<th>55</th>
<th>60</th>
<th>65</th>
<th>70</th>
<th>75</th>
<th>80</th>
<th>85</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>90</td>
<td>96</td>
<td>98</td>
<td>100</td>
<td>103</td>
<td>106</td>
<td>109</td>
<td>112</td>
<td>115</td>
<td>119</td>
</tr>
<tr>
<td>92</td>
<td>100</td>
<td>103</td>
<td>105</td>
<td>108</td>
<td>112</td>
<td>115</td>
<td>119</td>
<td>123</td>
<td>128</td>
</tr>
<tr>
<td>94</td>
<td>104</td>
<td>107</td>
<td>111</td>
<td>114</td>
<td>118</td>
<td>122</td>
<td>127</td>
<td>132</td>
<td>137</td>
</tr>
<tr>
<td>96</td>
<td>109</td>
<td>113</td>
<td>116</td>
<td>121</td>
<td>125</td>
<td>130</td>
<td>135</td>
<td>141</td>
<td>146</td>
</tr>
<tr>
<td>98</td>
<td>114</td>
<td>118</td>
<td>123</td>
<td>127</td>
<td>133</td>
<td>138</td>
<td>144</td>
<td>150</td>
<td>157</td>
</tr>
<tr>
<td>100</td>
<td>119</td>
<td>124</td>
<td>129</td>
<td>135</td>
<td>141</td>
<td>147</td>
<td>154</td>
<td>161</td>
<td>168</td>
</tr>
</tbody>
</table>

(a) Use the table to find a linear approximation of the heat index $I$ when the temperature is near $94^\circ$ and the relative humidity is near 60%.

\[
\begin{align*}
    \text{estimate } \frac{\Delta I}{\Delta T} & \approx \frac{116 - 105}{4} = \frac{11}{4} = 2.75 \\
    \text{estimate } \frac{\Delta I}{\Delta H} & \approx \frac{114 - 107}{10} = \frac{7}{10} = 0.7 \\
    \text{heat index } I & \approx 111 + 2.75(T - 94) + 0.7(H - 60)
\end{align*}
\]

(b) Estimate the heat index when the temperature is $95^\circ$ and the relative humidity is 58%.

\[
\begin{align*}
    T = 95^\circ & \Rightarrow \Delta T = 1^\circ \\
    H = 58\% & \Rightarrow \Delta H = -2\% \\
    \text{heat index } I & \approx 111 + (2.75)(1) - (0.7)(2) \\
    T = 95^\circ & \Rightarrow H = 58\% = 112.35^\circ
\end{align*}
\]
4. (20 points) Find the points on the hyperboloid of one sheet

\[ 4x^2 + y^2 - z^2 = 4 \]

where the tangent plane is parallel to the plane \(2x + 2y + z = 7\).

Let \( f(x, y, z) = 4x^2 + y^2 - z^2 \). Surface is the level set of level 4.

\( \nabla f = (8x)\hat{i} + (2y)\hat{j} + (-2z)\hat{k} \)

plane has normal vector \( \vec{N} = 2\hat{i} + 2\hat{j} + \hat{k} \)

Want points where \( \nabla f = \lambda \vec{N} \) for some scalar \( \lambda \).

\[ \Rightarrow 8x = 2\lambda \Rightarrow 4x = y = -2z \]

\[ 2y = 2\lambda \]

\[ -2z = \lambda \]

To find the points,

\[ 4x^2 + (4x)^2 - (-2x)^2 = 4 \]

\[ 16x^2 = 4 \Rightarrow x^2 = \frac{1}{4} \Rightarrow x = \pm \frac{1}{2} \]

Two points:

\( (\frac{1}{2}, 2, -1) \) and \( (-\frac{1}{2}, -2, 1) \)
5. (20 points) Find all critical points of the function \( f(x, y) = x^3 + y^3 + 6xy \) and classify them using the Second Partial Test.

\[
\begin{align*}
\frac{\partial f}{\partial x} &= 3x^2 + 6y = 0 \implies 3x^2 = -6y \\
\frac{\partial f}{\partial y} &= 3y^2 + 6x = 0 \implies 3y^2 = -6x \\
\end{align*}
\]

\[ x^2 = -2y \quad \text{and} \quad y^2 = -2x \]

\[ \implies x^4 = 4y^2 = 4(-2x) \]

\[ \implies x^4 + 8x = 0 \implies x(x^3 + 8) = 0 \]

\[ \implies x = 0 \quad \text{or} \quad x = -2 \]

\[ \implies (x, y) = (0, 0) \quad \text{or} \quad (x, y) = (-2, -2) \]

Hessian matrix: \[
\begin{pmatrix}
6 & 6 \\
6 & 6
\end{pmatrix}
\]

At \((x, y) = (0, 0)\), \( D = \det \begin{pmatrix} 6 & 6 \\ 6 & 6 \end{pmatrix} = -36 \) saddle

At \((x, y) = (-2, -2)\), \( D = \det \begin{pmatrix} -12 & 6 \\ 6 & -12 \end{pmatrix} = 108 \)

\[ \frac{\partial^2 f}{\partial x^2} < 0 \implies \text{local maximum} \]