Linear approximation

Last class we derived an equation for the plane tangent to the graph $z = f(x, y)$ at the point (a, b, c) where $c = f(a, b)$. It can be written as

$$z - c = \left(\frac{\partial f}{\partial x}(a, b)\right)(x - a) + \left(\frac{\partial f}{\partial y}(a, b)\right)(y - b).$$

This equation can also be thought of as a linear approximation to $f(x, y)$ for (x, y) near (a, b).

We can use the the formula for the tangent plane to define a "linear" function

$$L(x, y) = f(a, b) + \left(\frac{\partial f}{\partial x}(a, b)\right)(x - a) + \left(\frac{\partial f}{\partial y}(a, b)\right)(y - b).$$

The graph of this function is the tangent plane for $f(x, y)$ at the point (a, b), and it provides a linear approximation to $f(x, y)$ near (a, b).

Example. The linear approximation of the function $f(x, y) = 9 - x^2 - y^2$ near the point $(1, 2)$ is

$$f(x, y) \approx L(x, y) = 4 - 2(x - 1) - 4(y - 2).$$

Another way to write this approximation is as

$$f(1 + \Delta x, 2 + \Delta y) \approx 4 - 2\,\Delta x - 4\,\Delta y,$$

where $\Delta x = x - 1$ and $\Delta y = y - 2$.

Example. Calculate the linear approximation of the function $g(x, y) = y \ln(xy) + y$ near the point $(1/2, 2)$.

Second partials

Just as there is a second derivative for a function of one variable, there are four second partial derivatives for a function of two variables.

Example. Consider $g(x, y) = y \ln(xy) + y$ as discussed earlier. We have already calculated that

$$\frac{\partial g}{\partial x} = \frac{y}{x} \qquad \text{and} \qquad \frac{\partial g}{\partial y} = 2 + \ln(xy).$$

Consequently,

$$\frac{\partial^2 g}{\partial x^2} = -\frac{y}{x^2} \qquad \text{and} \qquad \frac{\partial^2 g}{\partial y^2} = \frac{1}{y}.$$

What about the other two partials

$$\frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right) \qquad \text{and} \qquad \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}\right)?$$

Clairaut's Theorem. If $f(x, y)$ and its partial derivatives

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y \partial x}$$

are continuous, then the order of partial differentiation is irrelevant. In other words,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

There is a link on the course web page to a discussion of an example for which the conclusion of Clairaut's Theorem does not hold. We will do our best to avoid such functions in this course.

Extreme values (max and min) for functions of two variables

We can use partial derivatives to help us find the extreme values—maxima and minima—of functions of more than one variable. Along the way, we also get some insight into what the second partials of a function measure.

Let's look at the today's map of high temperatures to see if we can get some idea of how this might be done. We always (just as in one-dimensional calculus) distinguish between absolute (or global) maxima and relative (or local) maxima. In the US, we often have many local minima of the temperature, but usually there is only one global minimum. Our techniques for finding such points are essentially the same as the ones that we used for functions on one variable. We look for critical points because every local minimum is a critical point.

First let's review what we did in one-variable calculus.

To find extrema of functions of more than one variable, we always look for the critical points first.

Definition. Let $z = f(x, y)$ be differentiable. If

$$\frac{\partial f}{\partial x}(a, b) = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y}(a, b) = 0,$$

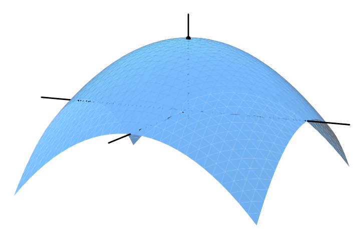
then (a, b) is a *critical point* of $f(x, y)$ and $c = f(a, b)$ is a *critical value*.

Note: The tangent plane to the graph $z = f(x, y)$ is horizontal at the critical point (a, b, c) because the normal vector

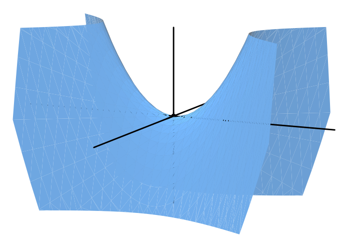
$$\mathbf{N} = \left[\frac{\partial f}{\partial x}(a, b)\right]\mathbf{i} + \left[\frac{\partial f}{\partial y}(a, b)\right]\mathbf{j} - \mathbf{k}$$

to the tangent plane reduces to $-\mathbf{k}$ at a critical point.

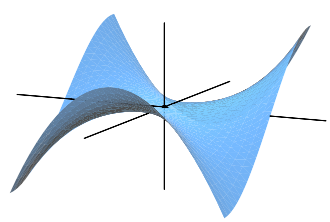
Example 1. Consider $f(x, y) = 9 - x^2 - y^2$.



Example 2. Consider $g(x, y) = y^2 - x^2$.



Example 3. Consider $h(x, y) = \frac{1}{3}x^3 - xy^2$.



Theorem. If $c = f(a,b)$ is an extreme value, then (a,b) is a critical point of f.

How do we determine the type of a critical point?

Second Partials Test: Let (a,b) be a critical point of $f(x,y)$. Form the *Hessian* matrix

$$H(a,b) = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

where $A = \dfrac{\partial^2 f}{\partial x^2}(a,b)$, $C = \dfrac{\partial^2 f}{\partial y^2}(a,b)$, and $B = \dfrac{\partial^2 f}{\partial x \partial y}(a,b) = \dfrac{\partial^2 f}{\partial y \partial x}(a,b)$. Let

$$D = \det H(a,b) = AC - B^2.$$

1. If $D > 0$, there are two cases.
   (a) If $A > 0$, then the critical point is a local minimum.
   (b) If $A < 0$, then the critical point is a local maximum.
2. If $D < 0$, then the critical point is a saddle.
3. If $D = 0$, the test is inconclusive.

Definition. If $D = 0$, we say that (a,b) is a *degenerate* critical point.

Let's return to the three examples discussed earlier.

Example 1. $f(x,y) = 9 - x^2 - y^2$

Example 2. $g(x,y) = y^2 - x^2$

Example 3. $h(x, y) = \frac{1}{3}x^3 - xy^2$

Warning. There are degenerate critical points that are extreme points. For example, the function

$$z = y^4 + x^2$$

has a degenerate critical point at $(0, 0)$, and it is an absolute minimum of the function.

Why does the Second Partials Test work the way that it does?

There is a proof of this test in Appendix E of our textbook, but I think that you would find the Discovery Project on p. 812 more informative. An important step in that project is step 4 where you use the technique of completing the square to understand the critical points of quadratic functions of the form

$$f(x, y) = ax^2 + bxy + cy^2$$

at $(0, 0)$.

When you look for extreme values of functions over bounded regions in the xy-plane, you need to analyze the critical points inside the region and analyze the behavior of the function along the boundary of the region. See pages 807 and 808 of your textbook for more details.