More on the multivariable chain rules

Chain Rule—Type I (continued)

Yesterday we learned that the multivariable chain rule for $f(P(t))$ is:

**Chain Rule.** The derivative of the composition $f(P(t))$ is given by

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$  

This version of the Chain Rule has an important formulation in terms of the gradient of $f$.

**Definition.** Given a function $f(x, y)$ that is differentiable at the point $(a, b)$. Then the *gradient vector of $f$ at $(a, b)$* is the vector

$$\nabla f(a, b) = \frac{\partial f}{\partial x}(a, b) \mathbf{i} + \frac{\partial f}{\partial y}(a, b) \mathbf{j}.$$  

Sometimes the gradient vector of $f$ is denoted $\text{grad} f(a, b)$.

**Restatement of the Chain Rule.** The derivative of the composition $f(P(t))$ is

$$\left.\frac{df}{dt}\right|_{t=t_0} = \nabla f(P(t_0)) \cdot P'(t_0).$$  

**Example.** Once again we return to $P(t) = t \mathbf{i} + t^2 \mathbf{j}$ and $f(x, y) = 2x^2 + y^2$. 
Animation of this chain rule

**Example.** Use the polar curve $r = \cos 2\theta$ to parametrize a curve $P(t)$ in the $xy$-plane and consider the composition $f(P(t))$ where

$$f(x, y) = y^2 - x^2.$$
This chain rule has some important theoretical implications as well.

**Theorem.**

1. Let \( f(x,y) \) be a differentiable function such that \( \nabla f(x,y) = 0 \) for all \((x,y)\). Then \( f(x,y) \) is a constant function.

2. If \( g(x,y) \) and \( h(x,y) \) are two differentiable functions such that

   \[
   \nabla g(x,y) = \nabla h(x,y)
   \]

   for all \((x,y)\). Then \( g(x,y) = h(x,y) + K \) for some constant \( K \).
Chain Rule—Type II

For this situation, consider a function \( f(x, y) \) of two variables and suppose that the variables \( x \) and \( y \) are functions of other variables.

For example, consider \( x \) and \( y \) as a function of the polar coordinates \( r \) and \( \theta \). That is,

\[
x = r \cos \theta \quad \text{and} \quad y = r \sin \theta.
\]

**Example.** Let \( f(x, y) = xy + y^2 \). What is the angular rate of change of \( f(x, y) \) at the point \((x, y) = (1, 2)\)?