Constrained max/min and the method of Lagrange multipliers

A nice application of the geometry of the gradient is the method of Lagrange multipliers. It is a method that locates extreme values subject to a constraint.

What is a constrained optimization problem? Here’s an example:

Suppose that we want to produce a cylindrical tin can, including a top and a bottom, using 100 cm$^2$ of tin, ignoring waste. What dimensions produce the can of maximum volume?

Before we tackle that problem, let’s consider more basic examples.

**Example 1.** Consider the function $f(x, y) = 2x^2 + 4y^2$. What are its extreme values if $x$ and $y$ are subject to the constraint $x^2 + y^2 = 1$?

**Example 2.** A somewhat easier example to analyze is the function $f(x, y) = \frac{1}{4}(y^2 - x^2)$ subject to the same constraint $x^2 + y^2 = 1$. Here are its level sets along with the constraint.
The method of Lagrange multipliers is based on the following theorem.

**Theorem.** The gradient $\nabla f$ is perpendicular to the constraint at the constrained max/min of $f$.

The method of Lagrange multipliers

The constraint is also a level set. That is, it is a level set of the constraint function $C$. If we combine the result of the theorem along with the fact that the gradient of $C$ is perpendicular to the level sets of $C$, we get the Lagrange multiplier equation

$$\nabla f(P) = \lambda \nabla C(P)$$

for some scalar $\lambda$ at points $P$ where the constrained max or min occurs.

**Example.** Back to Example 2: What points $(x, y)$ does the method of Lagrange multipliers identify?
Here are the gradient vectors of both $f(x, y) = \frac{1}{4}(y^2 - x^2)$ and $C(x, y) = x^2 + y^2$ along the constraint $x^2 + y^2 = 1$. The left-hand figure includes the gradient of $f(x, y)$, and the right-hand figure has the gradient of $C(x, y)$.

Here are both gradients in the same figure:
Example. Find the point on the plane $x + y + 2z = 1$ that is closest to the origin.
**Example.** A cylindrical tin can, including a top and a bottom, is to be manufactured using 100 cm$^2$ of tin, ignoring waste. What dimensions produce the can of maximum volume?