Constrained max/min and the method of Lagrange multipliers

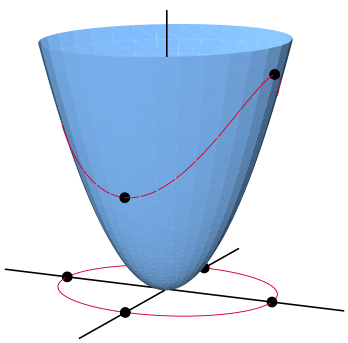
A nice application of the geometry of the gradient is the method of Lagrange multipliers. It is a method that locates extreme values *subject to a constraint*.

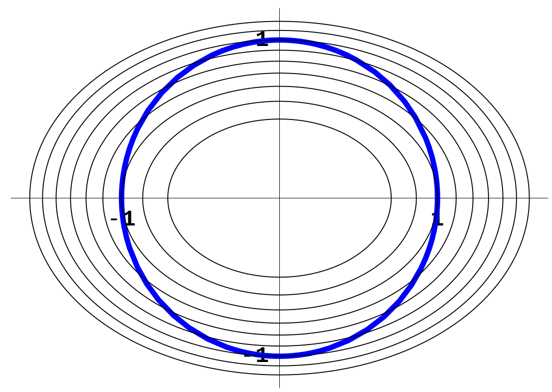
What is a constrained optimization problem? Here's an example:

Suppose that we want to produce a cylindrical tin can, including a top and a bottom, using 100 cm^2 of tin, ignoring waste. What dimensions produce the can of maximum volume?

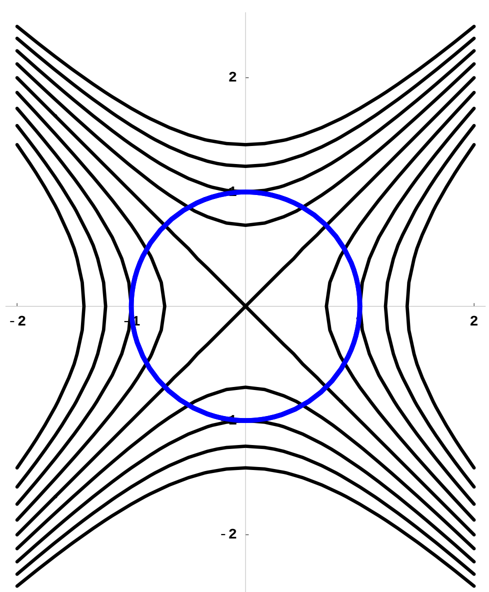
Before we tackle that problem, let's consider more basic examples.

Example 1. Consider the function $f(x, y) = 2x^2 + 4y^2$. What are its extreme values if x and y are subject to the constraint $x^2 + y^2 = 1$?





Example 2. A somewhat easier example to analyze is the function $f(x, y) = \frac{1}{4}(y^2 - x^2)$ subject to the same constraint $x^2 + y^2 = 1$. Here are its level sets along with the constraint.



The method of Lagrange multipliers is based on the following theorem.

Theorem. The gradient ∇f is perpendicular to the constraint at the constrained max/min of f.

The method of Lagrange multipliers

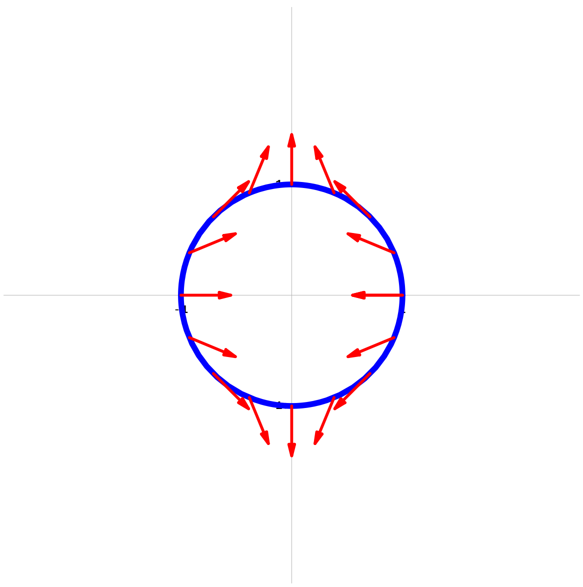
The constraint is also a level set. That is, it is a level set of the constraint function C. If we combine the result of the theorem along with the fact that the gradient of C is perpendicular to the level sets of C, we get the Lagrange multiplier equation

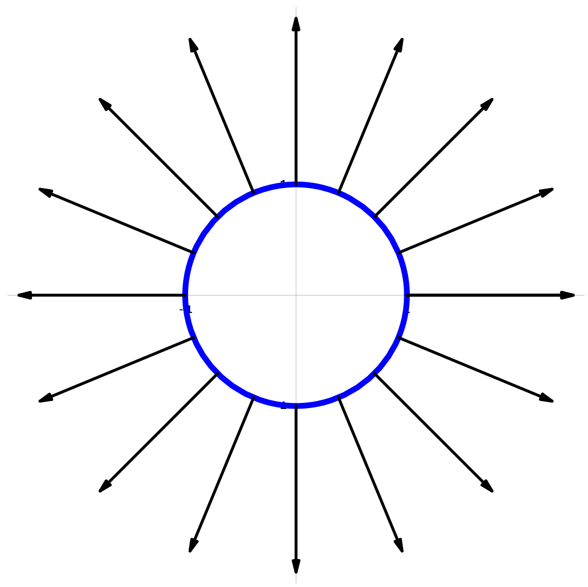
$$\nabla f(P) = \lambda \nabla C(P)$$

for some scalar λ at points P where the constrained max or min occurs.

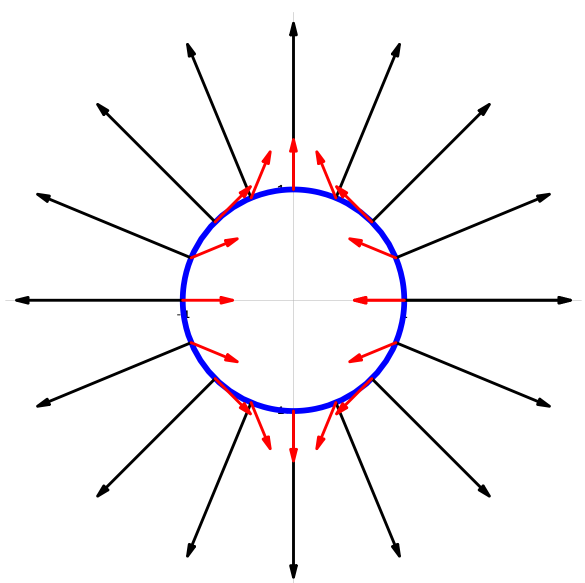
Example. Back to Example 2: What points (x, y) does the method of Lagrange multipliers identify?

Here are the gradient vectors of both $f(x, y) = \frac{1}{4}(y^2 - x^2)$ and $C(x, y) = x^2 + y^2$ along the constraint $x^2 + y^2 = 1$. The left-hand figure includes the gradient of $f(x, y)$, and the right-hand figure has the gradient of $C(x, y)$.





Here are both gradients in the same figure:



Example. Find the point on the plane $x + y + 2z = 1$ that is closest to the origin.

Example. A cylindrical tin can, including a top and a bottom, is to be manufactured using 100 cm^2 of tin, ignoring waste. What dimensions produce the can of maximum volume?