Double integrals over general regions

So far the double integrals that we have discussed have all been integrals over rectangles. Now we consider integrals over more general regions $R$.

We begin with a brief description about how such an integral is defined. There are basically two ways. Both involve enclosing the region in question inside a rectangle. Then one way involves making Riemann sums that only include subrectangles that lie entirely within the region in question. The other way involves integrating a new function $F(x, y)$ over the rectangle. The new function $F(x, y)$ is defined by

$$F(x, y) = \begin{cases} f(x, y), & \text{if } (x, y) \text{ is in } R; \\ 0, & \text{if } (x, y) \text{ is not in } R. \end{cases}$$

Either way produces the same result.

It is important that you remember the interpretations of the double integral that we discussed a few days ago. For example, recall that

$$\int_{R} f(x, y) \, dA = \text{(average value of } f(x, y) \text{ over } R)(\text{area } R).$$

Consequently, one way to compute the area of a region $R$ is by computing $\int_{R} 1 \, dA$.

Iterated integration for general regions

$x$-slices (Type I regions in your textbook)

Suppose that $R$ is a region that can be described as the region enclosed by two graphs of functions of $x$. 
For a Type I region,
\[ \iiint_{R} f(x, y) \, dA = \int_{a}^{b} \int_{b(x)}^{t(x)} f(x, y) \, dy \, dx. \]

**Example.** Let \( R \) be the region bounded by the line \( y = x \) and the graph of \( y = x^2 \). Calculate
\[ \iiint_{R} xy \, dA. \]
$y$-slices (Type II regions in your textbook)

Suppose that $R$ is a region that can be described as the region enclosed by two graphs of functions of $y$.

For a Type II region,

$$\int_R f(x, y) \, dA = \int_a^b \int_{g_1(y)}^{g_2(y)} f(x, y) \, dx \, dy.$$

**Example.** Let $R$ be the half-disk $\{(x, y) \mid x^2 + y^2 \leq 1, \ x \geq 0\}$. Calculate

$$\int_R x \, dA.$$ 

We can do this integral with either $x$-slices or $y$-slices.

First, let’s set up the integral using $x$-slices.
Now, let’s set up the same integral using $y$-slices.

What does $\iint_R x \, dA$ measure?
Sometimes the order in which you integrate is crucial.

**Example.** Consider the integral

\[
\int_0^1 \int_{3y}^3 e^{x^2} \, dx \, dy.
\]
Sometimes the region $R$ is neither a Type I region nor a Type II region.

**Example.** (Fall 2001 exam question) Consider the following region $R$ in the $xy$-plane.

It is bounded by the curves $y = x^2$ and $x = y^2 + 1$ and the lines $y = -1$, $y = 1$, and $x = -1$. Calculate

$$\iint_R x \, dA.$$ 

**Warning.** Avoid a couple of common mistakes. Note that:

1. The outside limits of integration must be constants.

2. The only variable that can appear in the inside limits of integration is the outside variable.