Double integrals over general regions

So far the double integrals that we have discussed have all been integrals over rectangles. Now we consider integrals over more general regions R.

We begin with a brief description about how such an integral is defined. There are basically two ways. Both involve enclosing the region in question inside a rectangle. Then one way involves making Riemann sums that only include subrectangles that lie entirely within the region in question. The other way involves integrating a new function $F(x, y)$ over the rectangle. The new function $F(x, y)$ is defined by

$$F(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \text{ is in } R; \\ 0, & \text{if } (x,y) \text{ is not in } R. \end{cases}$$

Either way produces the same result.

It is important that you remember the interpretations of the double integral that we discussed a few days ago. For example, recall that

$$\iint_R f(x,y)\,dA = (\text{average value of } f(x,y) \text{ over } R)(\text{area } R).$$

Consequently, one way to compute the area of a region R is by computing $\iint_R 1\,dA$.

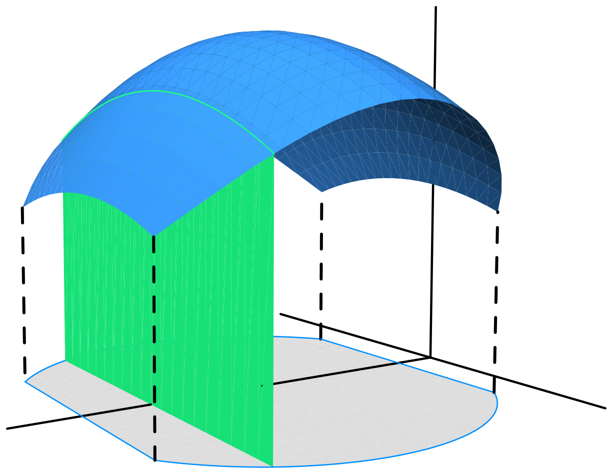
Iterated integration for general regions

x-slices (Type I regions in your textbook)

Suppose that R is a region that can be described as the region enclosed by two graphs of functions of x.

For a Type I region,

$$\iint\limits_R f(x,y)\,dA = \int_a^b \int_{b(x)}^{t(x)} f(x,y)\,dy\,dx.$$

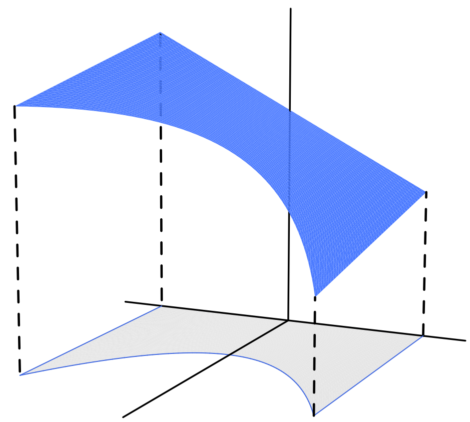


Example. Let R be the region bounded by the line $y = x$ and the graph of $y = x^2$. Calculate

$$\iint\limits_R xy\,dA.$$

y-slices (Type II regions in your textbook)

Suppose that R is a region that can be described as the region enclosed by two graphs of functions of y.



For a Type II region,

$$\iint_R f(x,y)\,dA = \int_c^d \int_{l(y)}^{r(y)} f(x,y)\,dx\,dy.$$

Example. Let R be the half-disk $\{(x,y) \mid x^2+y^2 \leq 1,\ x \geq 0\}$. Calculate

$$\iint_R x\,dA.$$

We can do this integral with either x-slices or y-slices.

First, let's set up the integral using x-slices.

Now, let's set up the same integral using y-slices.

What does $\iint\limits_R x\, dA$ measure?

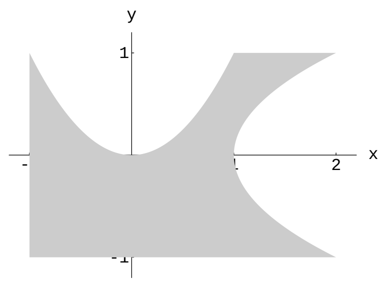
Sometimes the order in which you integrate is crucial.

Example. Consider the integral

$$\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy.$$

Sometimes the region R is neither a Type I region nor a Type II region.

Example. (Fall 2001 exam question) Consider the following region R in the xy-plane.



It is bounded by the curves $y = x^2$ and $x = y^2 + 1$ and the lines $y = -1$, $y = 1$, and $x = -1$. Calculate

$$\iint_R x \, dA.$$

Warning. Avoid a couple of common mistakes. Note that:

1. The outside limits of integration must be constants.
2. The only variable that can appear in the inside limits of integration is the outside variable.