More on partial derivatives

Last class we discussed the partial derivative of a function $f(x, y)$ with respect to x. We discussed both the definition and a shorthand way to compute it.

Let's try a more complicated example.

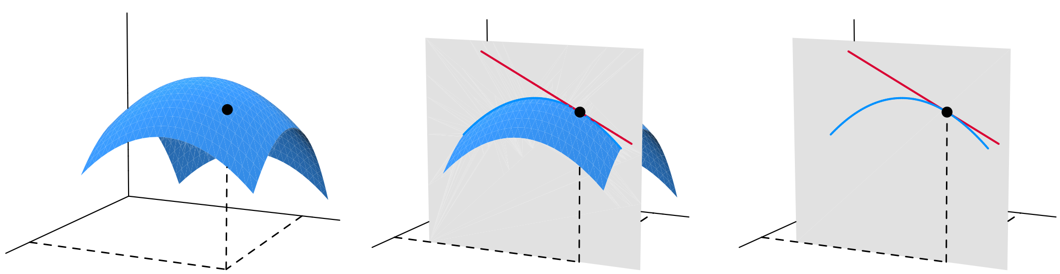
Example. Consider $g(x, y) = y \ln(xy) + y$.

The partial derivative with respect to y is defined in a similar fashion.

Definition. The partial derivative of $f(x, y)$ in the y-direction at the point (x_0, y_0) is defined by

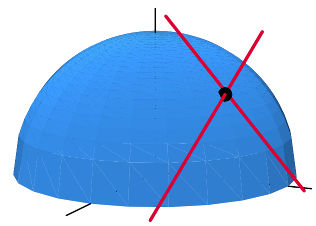
$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}.$$

We keep x constant and vary y as we take the limit.



Example. Consider $g(x, y) = y \ln(xy) + y$ again and calculate $\partial g / \partial y$ this time.

Example. Consider the function $f(x,y) = \sqrt{9-x^2-y^2}$ at the point $(1,2)$. In what direction, the x-direction or the y-direction, does $f(x,y)$ decrease most rapidly?



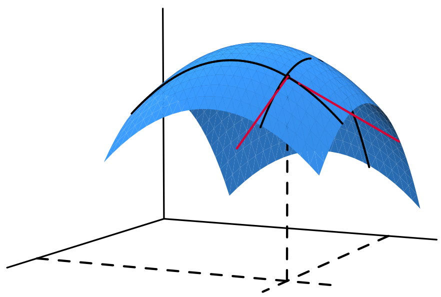
Geometric interpretation of partial derivatives

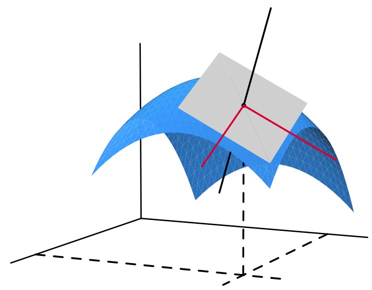
Now let's discuss the geometric significance of the two numbers that we obtain from the partials of $f(x,y) = 9-x^2-y^2$ at $(1,2)$. For example, we can use these numbers to calculate the equation of the tangent plane to the graph of $z = f(x,y)$ at the point $(1,2,4)$.

Definition. Suppose that the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist at the point (a, b). Then let

$$\mathbf{T}_x = \mathbf{i} + \left(\frac{\partial f}{\partial x}(a,b)\right)\mathbf{k} \qquad \text{and} \qquad \mathbf{T}_y = \mathbf{j} + \left(\frac{\partial f}{\partial y}(a,b)\right)\mathbf{k}.$$

The normal vector for $f(x, y)$ at the point (a, b) is $\mathbf{N} = \mathbf{T}_y \times \mathbf{T}_x$.





The equation for the tangent plane can be written as

$$z - c = \left(\frac{\partial f}{\partial x}(a,b)\right)(x-a) + \left(\frac{\partial f}{\partial y}(a,b)\right)(y-b),$$

where $c = f(a, b)$.

Linear approximation

The equation for the tangent plane can also be thought of as a linear approximation to $f(x,y)$ for (x,y) near (a,b).

We can use the the formula for the tangent plane to define a "linear" function

$$L(x,y) = f(a,b) + \left(\frac{\partial f}{\partial x}(a,b)\right)(x-a) + \left(\frac{\partial f}{\partial y}(a,b)\right)(y-b).$$

The graph of this function is the tangent plane for $f(x,y)$ at the point (a,b), and it provides a linear approximation to $f(x,y)$ near (a,b).

Example. The linear approximation of the function $f(x,y) = 9 - x^2 - y^2$ near the point $(1,2)$ is

$$f(x,y) \approx L(x,y) = 4 - 2(x-1) - 4(y-2).$$

Another way to write this approximation is as

$$f(1+\Delta x, 2+\Delta y) \approx 4 - 2\,\Delta x - 4\,\Delta y,$$

where $\Delta x = x - 1$ and $\Delta y = y - 2$.

Example. Calculate the linear approximation of the function $g(x,y) = y\ln(xy) + y$ near the point $(1/2, 2)$.