Triple integrals

Integrating functions of three variables is very similar to integrating functions of two variables. Intuitively,

$$\iiint_R f(x,y,z)\,dV = \begin{cases} 1. & \text{“sum” of } f(x,y,z) \text{ over } R \\ 2. & (\text{average value of } f(x,y,z) \text{ over } R)(\text{volume}(R)) \\ 3. & \text{four-dimensional “volume” under the graph of } f(x,y,z) \end{cases}$$

More precisely, the triple integral is a limit of Riemann sums. We partition the region R in space into small rectangular parallelopipids.

For each little “cube” R_{ijk}, we choose a point P_{ijk} in it and then we calculate the sum

$$S_{lmn} = \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(P_{ijk})(\Delta x)(\Delta y)(\Delta z).$$

As l, m, and $n \to \infty$, S_{lmn} approaches a limiting value

$$\iiint_R f(x,y,z)\,dV,$$

which we call the triple integral of the function $f(x,y,z)$ over the region R. This limit is independent of the manner in which we choose the “test” points P_{ijk}.

We calculate this limit by repeated integration just as we did to calculate double integrals.

Intuitively, triple integrals are no more complicated than double integrals, yet they are harder to compute because they are harder to set up. The difficulty comes when we try to set up these integrals over regions more general than rectangular boxes.

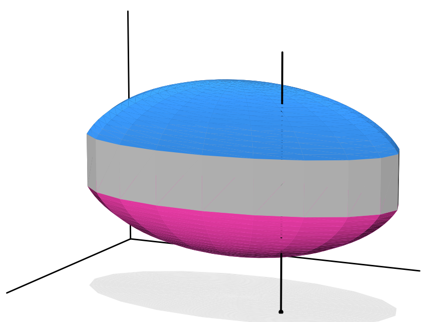
Example. Evaluate

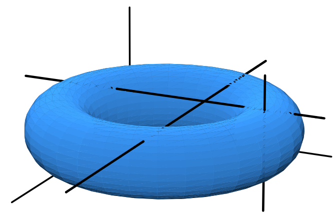
$$\iiint\limits_Q z\, dV$$

where Q is the region bounded by the cylinder $x^2 + z^2 = 9$, the plane $y + z = 3$, and the plane $y = 0$.

Definition. A solid region in space is z-simple if every vertical line that intersects the region enters and exits the region exactly once.

There are analogous definitions of x-simple and y-simple regions.





There is a version of Fubini's Theorem for triple integrals.

Theorem. (Fubini's Theorem) If Q is a z-simple region, then

$$\iiint_Q f(x,y,z)\,dV = \iint_{Q'} \left(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz \right) dA,$$

where Q' is the projection of Q onto the xy-plane and A is area in the xy-plane ($dA = dx\,dy$ or $dA = dy\,dx$).

Now back to the example at hand.

Example. Evaluate

$$\iiint_Q z\,dV$$

where Q is the region bounded by the cylinder $x^2 + z^2 = 9$, the plane $y + z = 3$, and the plane $y = 0$.

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We can also view this region as x-simple and evaluate the integral that way. Try setting up the integral where $dV = dx\,dz\,dy$. In other words,

$$\iiint_Q z\,dV = \int \int \quad \int \qquad z\,dx\,dz\,dy.$$

The answer is posted on the course web site.

Note that, in triple integrals,

1. the outermost limits of integration are constants,
2. the intermediate limits of integration only depend on the outermost variable, and
3. the intermost limits of integration only depend on the two outermost variables.