A little more about path integrals

At the end of last class, I quickly mentioned the following example.

Example. Consider a semicircular piece of wire of radius R. Find its center of mass. (Note that R is the radius of the semicircle, not the radius of the wire.)

If we think of the piece of wire as the semicircle

$$x^2 + y^2 = R^2, \quad y \geq 0,$$

Then $\overline{x} = 0$ and

$$\overline{y} = \frac{\int_C y\, ds}{\int_C ds} = \frac{\int_C y\, ds}{\pi R} = \left(\frac{2}{\pi}\right) R.$$

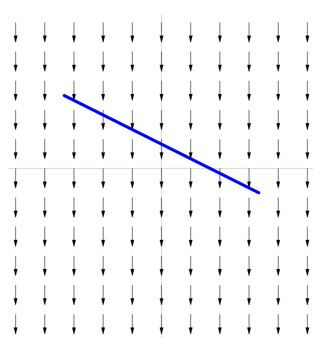
You should parametrize the semicircle and calculate the path integral to verify this result.

Line integrals

Last class we discussed path integrals. These are integrals with which we add up a scalar quantity over a curve in the plane or in space. Today we discuss an important special case of a path integral—the line integral.

Suppose that we have a vector field and a curve and we want to measure how much of the field is pointing in the direction of the curve. If the field is a gravitational field, then this quantity is the work done by the gravitational force in displacing a particle along the curve. If the field is a velocity field of a fluid, then this quantity is the circulation of the fluid along the curve. If the field is an electric field and the curve is closed, then this quantity is related to current passing through any surface bounded by the curve (Ampère's Law).

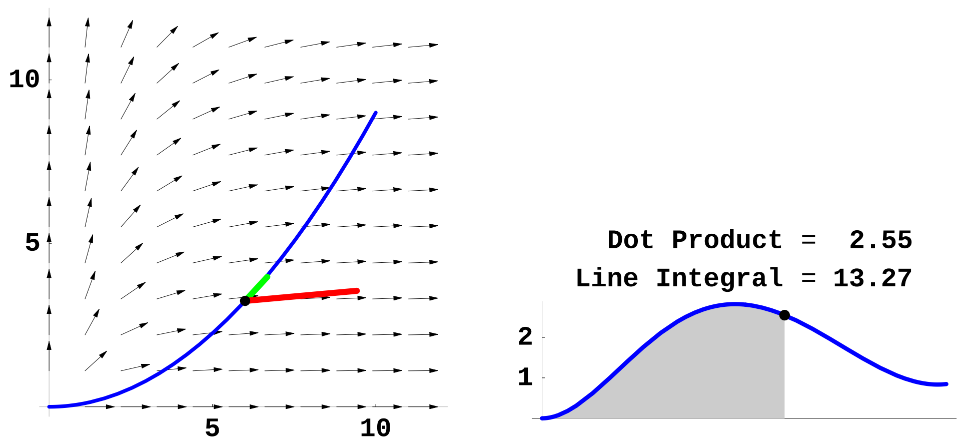
Example. Consider a constant force field $\mathbf{F}(x, y) = \mathbf{C}$ and a linear displacement $\mathbf{D}$.



Definition. Consider a vector field $\mathbf{F}(x, y)$ and a curve C. The line integral of $\mathbf{F}(x, y)$ along C is the path integral

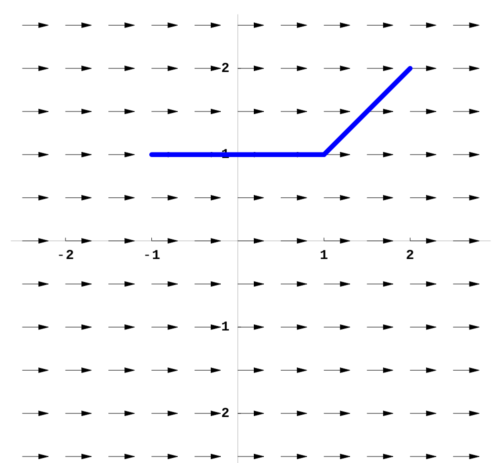
$$\int_C (\mathbf{F} \cdot \mathbf{T})\, ds,$$

where $\mathbf{T}$ is the unit tangent vector along the curve C.



Let's eyeball three examples before we start the computations.

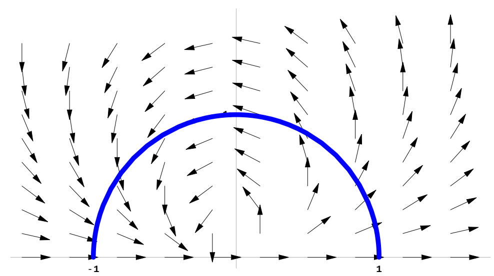
Example 1. Let $\mathbf{F}(x, y) = \mathbf{i}$ and C consists of two line segments—one from $(-1, 1)$ to $(1, 1)$ and another from $(1, 1)$ to $(2, 2)$.



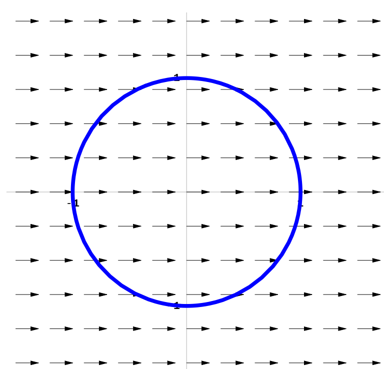
Example 2. Let

$$\mathbf{F}(x, y) = (x^2 - y^2)\mathbf{i} + 2xy\mathbf{j}$$

and C consists of the top half of the positively-oriented unit circle centered at the origin.



Example 3. Let $\mathbf{F}(x, y) = \mathbf{i}$ and C be the positively-oriented unit circle centered at the origin.



How do we go about calculating a line integral?

Suppose that the curve C is parametrized by a vector-valued function $\mathbf{r}(t)$ from $t = a$ to $t = b$. Then

$$\begin{aligned}\int_C (\mathbf{F} \cdot \mathbf{T})\, ds &= \int_a^b \left(\mathbf{F} \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\right) |\mathbf{r}'(t)|\, dt \\ &= \int_a^b (\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t))\, dt \\ &\equiv \int_C \mathbf{F} \cdot d\mathbf{r}.\end{aligned}$$

Example. What integral do we have to calculate to determine the value of the line integral in Example 2 above?