More on gradient vector fields

Last class we learned that line integrals are easier to compute if the vector field in question is the gradient of a function.

Theorem. (Fundamental Theorem of Calculus for line integrals) If $\mathbf{F} = \nabla f$, then

$$\int_{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

for any vector-valued function $\mathbf{r}(t)$ defined on the interval $a \leq t \leq b$.

Note that this theorem implies that the value of the line integral in this case depends only on the endpoints of the curve and not on the curve itself. This is called *path independence.*

Consequences of the Fundamental Theorem for line integrals:

If $\mathbf{F} = \nabla f$, then we have

1. path independence, and
2. the line integral $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every *closed* curve.

Theorem. If $\mathbf{F}$ satisfies either item 1 or item 2, then $\mathbf{F} = \nabla f$ for some function f.

Terminology: If $\mathbf{F}$ is a force field and any one of these three equivalent situations occur, then f is potential energy, and we have conservation of energy for this vector field. In this case, we say that the vector field is *conservative.*

Example. Let $\mathbf{x} = (x, y, z)$ and consider the gravitational force field

$$\mathbf{G}(\mathbf{x}) = -\frac{MG}{|\mathbf{x}|^3}\mathbf{x}$$

that we discussed on November 14. Then $\mathbf{G}(\mathbf{x}) = \nabla g(\mathbf{x})$, where $g(\mathbf{x}) = \dfrac{MG}{|\mathbf{x}|}$. We say that g is a potential function for $\mathbf{G}$.

So how can we tell if a given vector field is the gradient of a function? In other words, how can we tell if a vector field is conservative, and how can we construct a potential function if it is?

Necessary condition:

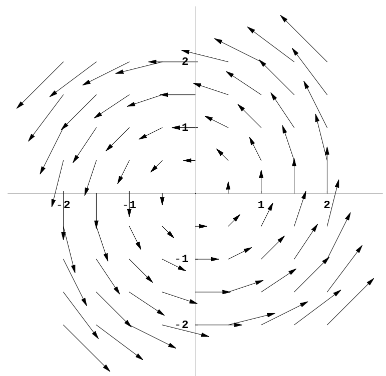
Suppose that $\mathbf{F}(x, y) = \nabla f(x, y)$, that is, suppose

$$\mathbf{F}(x, y) = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j}.$$

Necessary condition: If the vector field $\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$, then $\mathbf{F}(x,y)$ has a potential function only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Example. Consider the merry-go-round vector field $\mathbf{F}(x,y) = -y\mathbf{i} + x\mathbf{j}$.



Sufficient condition: Is the condition

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

enough to guarantee that $\mathbf{F}(x,y)$ has a potential function? Sometimes ...

Theorem. Suppose that $P(x,y)$ and $Q(x,y)$ are defined and continuously differentiable for all (x,y). If

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

for all (x,y), then $\mathbf{F}(x,y)$ has a potential function.

How do we find $f(x, y)$?

Example. Consider the vector field $\mathbf{F}(x, y) = (1 + e^y)\ \mathbf{i} + (xe^y + y^2)\ \mathbf{j}$.

Example. Consider the vector field $\mathbf{F}(x, y) = (x^2 + y)\ \mathbf{i} + (\sin x + y^2)\ \mathbf{j}$.

Green's Theorem

Green's Theorem relates line integrals of vector fields in the xy-plane to double integrals.

What is a positively-oriented, simple, closed curve in the plane?

Theorem. (Green's Theorem) Let C be a positively-oriented, simple, closed curve in the plane and let D denote the region it encloses. Then

$$\oint_C P\,dx + Q\,dy = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Example. Let C be the perimeter of the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$. Calculate

$$\oint_C x\,dx + xy\,dy.$$

Note: If $\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$ has a potential function, then

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0.$$