More on Green’s Theorem

Green’s Theorem relates line integrals of vector fields in the $xy$-plane to double integrals.

**Theorem.** (Green’s Theorem) Let $C$ be a positively-oriented, simple, closed curve in the plane and let $D$ denote the region it encloses. Then

$$\oint_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA.$$  

**Example.** Let $C$ be the perimeter of the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$. Calculate

$$\oint_C x \, dx + xy \, dy.$$  

**Note:** If $\mathbf{F}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}$ has a potential function, then

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0,$$

and we see that $\oint_C P \, dx + Q \, dy = 0.$
Example. Compute the line integral

$$\int -y^3 \, dx + x^3 \, dy$$

over the unit circle in the positively-oriented direction.
The curl of a planar vector field

I would like to use Green’s Theorem to explain one of the basic concepts in vector analysis—the curl of a vector field—in the case where the vector field $\mathbf{F}$ is a planar vector field. It helps if you consider the vector field $\mathbf{F}$ as a velocity field of a fluid and you imagine a little “paddle wheel” suspended in the fluid (see Figure 6 on p. 963 of your text). We would like to measure how much the paddle wheel rotates as it moves through the fluid.

For velocity fields of fluids, the line integral $\int_C \mathbf{F} \cdot \mathbf{T} \, ds$ is called the circulation of the fluid along the curve. The angular velocity of the paddle wheel at the point $(x_0, y_0)$ is one-half of the circulation density of the velocity field at the point $(x_0, y_0)$. Circulation density is defined to be the limit

$$\lim_{r \to 0} \frac{\int_C \mathbf{F} \cdot \mathbf{T} \, ds}{\text{area inside } C}$$

where $C$ is a circle of radius $r$ centered at $(x_0, y_0)$.

Here are five examples to illustrate this relationship.

**Example 1.** Let $\mathbf{F}(x, y) = \mathbf{i}$.

**Example 2.** Let $\mathbf{F}(x, y) = xi + yj$. 
Example 3. Let $F(x, y) = -yi + xj$.

Example 4. Let $F(x, y) = yi$.

Example 5. Let $F(x, y) = -y^2 i + x^2 j$. 
**Definition.** For a planar vector field \( \mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} \), the curl of \( \mathbf{F}(x, y) \) is the vector field

\[
\text{curl } \mathbf{F}(x, y) = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}.
\]

To interpret the curl of \( \mathbf{F} \) in this situation, we use Green’s Theorem.

**Theorem.** (The vector form of Green’s Theorem) Let \( C \) be a positively-oriented, simple, closed curve in the \( xy \)-plane and let \( D \) be the region that is enclosed by \( C \). Then

\[
\oint_C \mathbf{F} \cdot \mathbf{T} \, ds = \iint_D (\text{curl } \mathbf{F}) \cdot \mathbf{k} \, dA.
\]

How does this help us interpret the curl of \( \mathbf{F} \)?