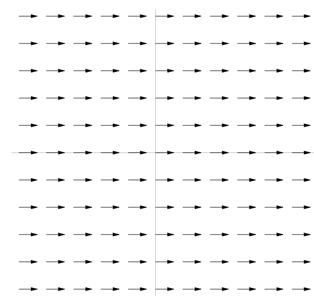
Divergence for planar vector fields

Definition. Given a vector field in the plane $\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$, the divergence of $\mathbf{F}$ is the scalar field (scalar function) defined by

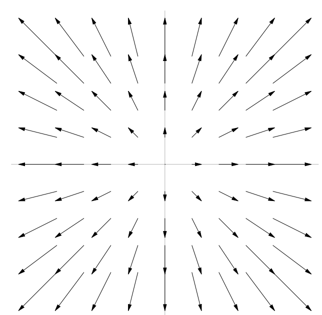
$$\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

Here are three examples.

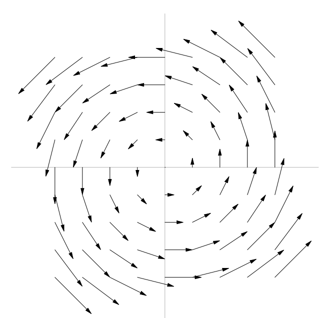
Example 1. Let $\mathbf{F}(x,y) = \mathbf{i}$.



Example 2. Let $\mathbf{F}(x,y) = x\mathbf{i} + y\mathbf{j}$.



Example 3. Let $\mathbf{F}(x,y) = -y\mathbf{i} + x\mathbf{j}$.



To understand what divergence measures in the case of our velocity field of a fluid, we consider a different path integral. Given a simple, closed curve C in the plane, consider the path integral

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds,$$

where $\mathbf{n}$ is a unit normal vector to C that points outside the region enclosed by C.

To see that this path integral is also a line integral, we need to recall two facts from earlier in the semester:

1. Suppose that the curve C is parametrized by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. Then we can reparametrize C using arc length s as the parameter (see pp. 709–710 in our text). Then the unit tangent vector

$$\mathbf{T} = \left(\frac{dx}{ds}\right)\mathbf{i} + \left(\frac{dy}{ds}\right)\mathbf{j}.$$

2. Because the curve C is positively oriented, we get the unit normal vector $\mathbf{n}$ by rotating $\mathbf{T}$ by $\pi/2$ radians in the clockwise direction. Given any vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ in the plane, then the vector perpendicular to $\mathbf{u}$ rotated by $\pi/2$ radians in the clockwise direction is $u_2\mathbf{i} - u_1\mathbf{j}$.

Now we use these two facts to convert $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$ to a line integral, and then we apply Green's Theorem to the result.

Theorem. (planar Divergence Theorem) Let C be a positively-oriented, simple, closed curve in the xy-plane and let D be the region that is enclosed by C. Then

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint\limits_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right)\, dA = \iint\limits_D (\operatorname{div} \mathbf{F})\, dA.$$

This identity justifies the name "divergence."

Curl and divergence of vector fields in space

From our discussion of curl and divergence for planar vector fields associated with fluid flow, you know that the curl of a vector field measures how a tiny paddle wheel rotates as it moves through the fluid. The divergence of a vector field measures how the fluid "expands." Now let's see how these concepts are defined for vector fields in space.

Definition. Given a vector field in space

$$\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k},$$

the curl of $\mathbf{F}$ is another vector field defined by

$$\operatorname{curl} \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right)\, \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right)\, \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\, \mathbf{k}.$$

This formula seems quite nasty, but there is an easy way to remember it.

$$\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x,y,z) & Q(x,y,z) & R(x,y,z) \end{vmatrix}.$$

Example. Calculate $\operatorname{curl}\mathbf{F}$ for $\mathbf{F}(x,y,z) = x^2y\,\mathbf{i} + yz^2\,\mathbf{j} + x^2z\,\mathbf{k}$.

There is an animation of the concept of curl from the University of Minnesota that is referenced on the class web site.

The formula for the curl is related to the question of whether a vector field in space has a potential function.

Theorem. If the vector field $\mathbf{F}(x,y,z)$ in space is the gradient of some function $f(x,y,z)$, then

$$\operatorname{curl}\mathbf{F} = \mathbf{0}.$$

Sometimes you will see this fact expressed very succinctly as

$$\nabla\times\nabla f = \mathbf{0}.$$

The theorem gives a necessary condition for a vector field in space to have a potential function. As in the case of vector fields in the plane, this necessary condition is also sufficient if the vector field is continuously differentiable everywhere.

Theorem. Suppose the vector field $\mathbf{F}$ is defined and continuously differentiable for all (x, y, z). If

$$\operatorname{curl} \mathbf{F} = \mathbf{0}$$

for all (x, y, z), then $\mathbf{F}$ has a potential function.

The potential function is calculated using a generalization of the procedure that we discussed on November 24.

We turn to the concept of divergence for vector fields in space.

Definition. Given a vector field in space

$$\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k},$$

the divergence of $\mathbf{F}$ is the scalar field (scalar function) defined by

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Shorthand notation: $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}$.

Example. Calculate $\operatorname{div} \mathbf{F}$ for $\mathbf{F}(x, y, z) = x^2 y\,\mathbf{i} + yz^2\,\mathbf{j} + x^2 z\,\mathbf{k}$.