More on curl and divergence of vector fields in space

Definition. Given a vector field in space

$$\mathbf{F}(x,y,z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k},$$

the curl of $\mathbf{F}$ is another vector field defined by

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

A good way to remember this definition involves the "del" operator ∇.

$$\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x,y,z) & Q(x,y,z) & R(x,y,z) \end{vmatrix}.$$

The formula for the curl is related to the question of whether a vector field in space has a potential function.

Theorem. If the vector field $\mathbf{F}(x,y,z)$ in space is the gradient of some function $f(x,y,z)$, then

$$\operatorname{curl}\mathbf{F} = \mathbf{0}.$$

Sometimes you will see this fact expressed very succinctly as

$$\nabla\times\nabla f = \mathbf{0}.$$

The theorem gives a necessary condition for a vector field in space to have a potential function. As in the case of vector fields in the plane, this necessary condition is also sufficient if the vector field is continuously differentiable everywhere.

Theorem. Suppose the vector field $\mathbf{F}$ is defined and continuously differentiable for all (x,y,z). If

$$\operatorname{curl}\mathbf{F} = \mathbf{0}$$

for all (x,y,z), then $\mathbf{F}$ has a potential function.

The potential function is calculated using a generalization of the procedure that we discussed on November 24.

We turn to the concept of divergence for vector fields in space.

Definition. Given a vector field in space

$$\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k},$$

the divergence of $\mathbf{F}$ is the scalar field (scalar function) defined by

$$\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Shorthand notation: $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F}$.

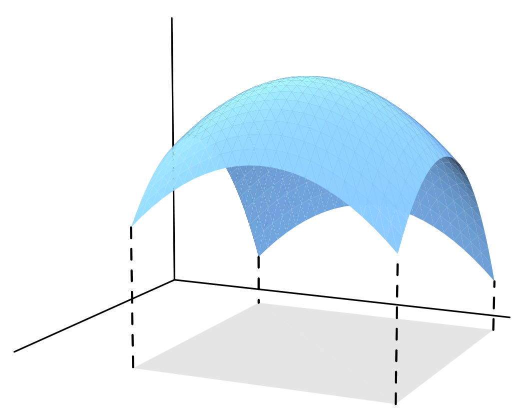
Example. Calculate $\operatorname{div}\mathbf{F}$ for $\mathbf{F}(x, y, z) = x^2y\,\mathbf{i} + yz^2\,\mathbf{j} + x^2z\,\mathbf{k}$.

Surface integrals

A path integral is how we "add up" a function over a curve in the plane or in space. Similarly, a surface integral is how we add up a function $f(x, y, z)$ over a surface in space.

We want an integral such that

$$\begin{aligned}\iint_S f(x, y, z)\,dS &= \text{ the "sum" of } f(x, y, z) \text{ over the surface } S\\ &= (\text{the average of } f(x, y, z) \text{ over } S)(\text{surface area}(S)).\end{aligned}$$



Special case: Assume that the surface S is the graph of a function $g(x, y)$ over a region R in the xy-plane. In this case, the differential of surface area is

$$dS = \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\; dA.$$

This differential can also be expressed as $dS = |\mathbf{N}|\, dA$, where the vector

$$\mathbf{N} = \left(\frac{\partial g}{\partial x}\right)\mathbf{i} + \left(\frac{\partial g}{\partial y}\right)\mathbf{j} - \mathbf{k}$$

is the normal vector to the surface (see the handout for November 3.).

Consequently, we have

$$\iint_S f(x, y, z)\, dS = \iint_R f(x, y, g(x, y)) \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\; dA.$$

Example. Let S be the portion of the cone $z^2 = x^2 + y^2$ that lies between the planes $z = 1$ and $z = 2$. Let's find its center of mass.
If the center of mass is denoted $(\overline{x}, \overline{y}, \overline{z})$, then we know that $\overline{x} = \overline{y} = 0$. Also,

$$\overline{z} = \frac{\displaystyle\iint_S z\, dS}{\text{area}(S)}.$$

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