More on curl and divergence of vector fields in space

**Definition.** Given a vector field in space

\[ \mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}, \]

the curl of \( \mathbf{F} \) is another vector field defined by

\[ \text{curl} \, \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x, y, z) & Q(x, y, z) & R(x, y, z) \end{vmatrix}. \]

A good way to remember this definition involves the “del” operator \( \nabla \).

The formula for the curl is related to the question of whether a vector field in space has a potential function.

**Theorem.** If the vector field \( \mathbf{F}(x, y, z) \) in space is the gradient of some function \( f(x, y, z) \), then

\[ \text{curl} \, \mathbf{F} = \mathbf{0}. \]

Sometimes you will see this fact expressed very succinctly as

\[ \nabla \times \nabla f = \mathbf{0}. \]

The theorem gives a necessary condition for a vector field in space to have a potential function. As in the case of vector fields in the plane, this necessary condition is also sufficient if the vector field is continuously differentiable everywhere.

**Theorem.** Suppose the vector field \( \mathbf{F} \) is defined and continuously differentiable for all \((x, y, z)\). If

\[ \text{curl} \, \mathbf{F} = \mathbf{0} \]

for all \((x, y, z)\), then \( \mathbf{F} \) has a potential function.

The potential function is calculated using a generalization of the procedure that we discussed on November 24.
We turn to the concept of divergence for vector fields in space.

**Definition.** Given a vector field in space
\[ \mathbf{F}(x, y, z) = P(x, y, z) \mathbf{i} + Q(x, y, z) \mathbf{j} + R(x, y, z) \mathbf{k}, \]
the divergence of \( \mathbf{F} \) is the scalar field (scalar function) defined by
\[ \text{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}. \]

Shorthand notation: \( \text{div} \mathbf{F} = \nabla \cdot \mathbf{F} \).

**Example.** Calculate \( \text{div} \mathbf{F} \) for \( \mathbf{F}(x, y, z) = x^2y \mathbf{i} + yz^2 \mathbf{j} + x^2z \mathbf{k} \).

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Surface integrals

A path integral is how we “add up” a function over a curve in the plane or in space. Similarly, a surface integral is how we add up a function \( f(x, y, z) \) over a surface in space.

We want an integral such that
\[ \iint_S f(x, y, z) \, dS = \text{the “sum” of } f(x, y, z) \text{ over the surface } S \]
\[ = (\text{the average of } f(x, y, z) \text{ over } S)(\text{surface area}(S)). \]
Special case: Assume that the surface \( S \) is the graph of a function \( g(x, y) \) over a region \( R \) in the \( xy\)-plane. In this case, the differential of surface area is

\[
dS = \sqrt{1 + \left( \frac{\partial g}{\partial x} \right)^2 + \left( \frac{\partial g}{\partial y} \right)^2} \, dA.
\]

This differential can also be expressed as \( dS = |\mathbf{N}| \, dA \), where the vector

\[
\mathbf{N} = \left( \frac{\partial g}{\partial x} \right) \mathbf{i} + \left( \frac{\partial g}{\partial y} \right) \mathbf{j} - \mathbf{k}
\]

is the normal vector to the surface (see the handout for November 3.).

Consequently, we have

\[
\iint_S f(x, y, z) \, dS = \iint_R f(x, y, g(x, y)) \sqrt{1 + \left( \frac{\partial g}{\partial x} \right)^2 + \left( \frac{\partial g}{\partial y} \right)^2} \, dA.
\]

**Example.** Let \( S \) be the portion of the cone \( z^2 = x^2 + y^2 \) that lies between the planes \( z = 1 \) and \( z = 2 \). Let’s find its center of mass.

If the center of mass is denoted \( (\bar{x}, \bar{y}, \bar{z}) \), then we know that \( \bar{x} = \bar{y} = 0 \). Also,

\[
\bar{z} = \frac{\iint_S z \, dS}{\text{area}(S)}.
\]