More about lines

Last class we used vector addition and scalar multiplication to describe a line in space. Given two points $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ on the line, we first calculate a direction vector $\mathbf{D}$ for the line. It is

$$\mathbf{D} = (b_1 - a_1)\mathbf{i} + (b_2 - a_2)\mathbf{j} + (b_3 - a_3)\mathbf{k}.$$  

If $\mathbf{P}_1$ is a position vector representing any point on the line, then a (position) vector equation for the line is

$$\mathbf{L}(t) = \mathbf{P}_1 + t\mathbf{D}.$$  

**Example.** Find a vector equation for the line $\ell$ that contains the points $(1,1,2)$ and $(3,7,-2)$.

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Summary: A direction vector for \( \ell \) is \( \mathbf{D} = 2 \mathbf{i} + 6 \mathbf{j} - 4 \mathbf{k} \).

Vector form for \( \ell \):

\[
\mathbf{L}(t) = (1 + 2t) \mathbf{i} + (1 + 6t) \mathbf{j} + (2 - 4t) \mathbf{k}
\]

Parametric form for \( \ell \):

\[
\begin{align*}
x(t) &= 1 + 2t \\
y(t) &= 1 + 6t \\
z(t) &= 2 - 4t
\end{align*}
\]

Symmetric form for \( \ell \):

\[
\frac{x - 1}{2} = \frac{y - 1}{6} = \frac{z - 2}{-4}
\]

Here’s another example that has a slightly different symmetric form. Try it on your own. (See the bottom of page 667 in your textbook for additional information.)

**Example.** Find an equation of the line through \((2, 4, 6)\) and \((1, 6, 6)\).
Dot Product

Given two vectors \( \mathbf{A} \) and \( \mathbf{B} \), their **dot product** is defined to be

\[
\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}| \cdot |\mathbf{B}| \cos \theta,
\]

where \( \theta \) is the angle between \( \mathbf{A} \) and \( \mathbf{B} \).

Important: Note that we start with two vectors and end up with a scalar. Therefore, we cannot take the dot product of three vectors in a row. Before we do a few examples, I would like to make a few observations.

1. Suppose \( |\mathbf{A}|, \ |\mathbf{B}| \neq 0 \). Then the two vectors \( \mathbf{A} \) and \( \mathbf{B} \) are perpendicular (orthogonal) \( \iff \mathbf{A} \cdot \mathbf{B} = 0 \).

2. There is a close relationship between projections and the dot product.

\[
\text{comp}_\mathbf{A}(\mathbf{B}) = |\mathbf{B}| \cos \theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}|} \quad \text{and} \quad \text{proj}_\mathbf{A}(\mathbf{B}) = (|\mathbf{B}| \cos \theta) \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{(\mathbf{A} \cdot \mathbf{B})}{|\mathbf{A}|^2} \mathbf{A}
\]

3. We can derive an algebraic formula for the dot product using the law of cosines. Let \( \mathbf{C} = \mathbf{B} - \mathbf{A} \).

The law of cosines is \( |\mathbf{C}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2 |\mathbf{A}| |\mathbf{B}| \cos \theta \). Therefore,

\[
2(\mathbf{A} \cdot \mathbf{B}) = |\mathbf{A}|^2 + |\mathbf{B}|^2 - |\mathbf{C}|^2
= a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - (b_1 - a_1)^2 - (b_2 - a_2)^2 - (b_3 - a_3)^2.
\]
So \(2(A \cdot B) = 2(a_1 b_1 + a_2 b_2 + a_3 b_3)\), and therefore,
\[
A \cdot B = a_1 b_1 + a_2 b_2 + a_3 b_3.
\]
This equation gives us an algebraic formula with geometric applications.

**Examples.** Let \(A = 2i + j + k\) and \(B = 3i + 4j + 3k\).

1. Compute the angle between \(A\) and \(B\).

2. Compute the length of the projection of \(A\) in the direction of \(B\).

Now let’s use the dot product to derive the triangle inequality. Recall that
\[
|A|^2 = a_1^2 + a_2^2 + a_3^2,
\]
so
\[
|A|^2 = A \cdot A.
\]
We can therefore derive
\[
|A + B|^2 = (A + B) \cdot (A + B)
\]
\[
= A \cdot A + 2(A \cdot B) + B \cdot B
\]
\[
= |A|^2 + 2|A||B| \cos \theta + |B|^2
\]
\[
\leq |A|^2 + 2|A||B| + |B|^2
\]
\[
\leq (|A| + |B|)^2
\]
and arrive at triangle inequality

\[ |\mathbf{A} + \mathbf{B}| \leq |\mathbf{A}| + |\mathbf{B}|. \]

The length function \(|\mathbf{A}|\) has four important properties:

1. \(|\mathbf{A}| \geq 0\) for all \(\mathbf{A}\)
2. \(|\mathbf{A}| = 0 \iff \mathbf{A} = \mathbf{0}\)
3. \(|r\mathbf{A}| = |r||\mathbf{A}|\)
4. \(|\mathbf{A} + \mathbf{B}| \leq |\mathbf{A}| + |\mathbf{B}|\)

The dot product has the following properties:

1. \(\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}\)
2. \(\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2\)
3. \(r(\mathbf{A} \cdot \mathbf{B}) = (r\mathbf{A}) \cdot \mathbf{B} = \mathbf{A} \cdot r\mathbf{B}\)
4. \(\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}\)
5. \(|\mathbf{A} \cdot \mathbf{B}| \leq |\mathbf{A}||\mathbf{B}| \) (which follows from \(\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta\))

Now let’s apply what we have learned to a problem involving lines.

**Example.** Let \(\ell_1\) be line through \((1,0,0)\) and \((1,2,2)\). Let \(\ell_2\) be the line through \((1,1,1)\) and \((1 + \sqrt{2},2,2)\). Do these lines intersect? What is the angle of intersection?