MA 226 Sample Examinations

Attached are the four examinations that were given in my MA 226 class in the Spring of 2003. You should use them to get an idea of the format of a typical test and to see the types of questions I ask. *You should not assume that the test questions this semester will be on the same topics.* In fact, you are always responsible for *all* of the material that we cover in class as well as *all* of the designated material from your text, and the best way to study for my exams is to be sure that you are very comfortable with the homework assignments and the examples that I present in class. My tests often vary in difficulty (as you can see from the attached), and your grade for the examination will be determined by a curve that will be announced in class after the examination is graded.

MA 226	Exam 1A	February 12, 2003

Name: ______________________________ Last five digits of ID number: __________

Discussion Section (circle yours): M 12–1 M 2–3 M 3–4 T 2:30–3:30 T 3:30–4:30

Directions: Please do all of your work in this exam booklet and make sure that you cross out any work that we should ignore when we grade. **Books and extra papers are not permitted.** If you have a question about a problem, please ask. Remember: answers that are written logically and clearly will receive higher scores. There are 5 questions on 6 pages (not counting this cover page). Please make sure that you have all 6 pages of questions.

Do not write in the following box:

PROBLEM	POSSIBLE	SCORE
Name, etc.	2	
1	24	
2	20	
3	22	
4	20	
5	12	
TOTAL	100	

1. (24 points) Consider the following 8 first-order equations:

1. $\dfrac{dy}{dt} = ty + t$	2. $\dfrac{dy}{dt} = y^2 + 1$	3. $\dfrac{dy}{dt} = ty - t$	4. $\dfrac{dy}{dt} = \sin t$
5. $\dfrac{dy}{dt} = y - t^2$	6. $\dfrac{dy}{dt} = \cos t$	7. $\dfrac{dy}{dt} = y + t^2$	8. $\dfrac{dy}{dt} = 1 - y^2$

Four of the associated slope fields are shown on the next page. Pair the slope fields with their associated equations. Provide a brief justification for your choice. **You will not receive any credit unless you justify your selection.**

(a) The equation for slope field A is ____. My reason for choosing this answer is:

(b) The equation for slope field B is ____. My reason for choosing this answer is:

(c) The equation for slope field C is ____. My reason for choosing this answer is:

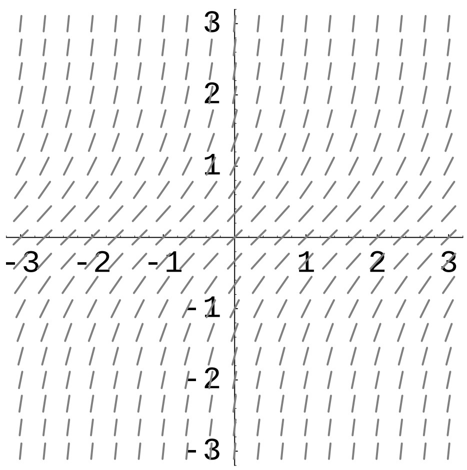
(d) The equation for slope field D is ____. My reason for choosing this answer is:

1. (continued) **Answer this question on the previous page.** The equations are provided here for your convenience:

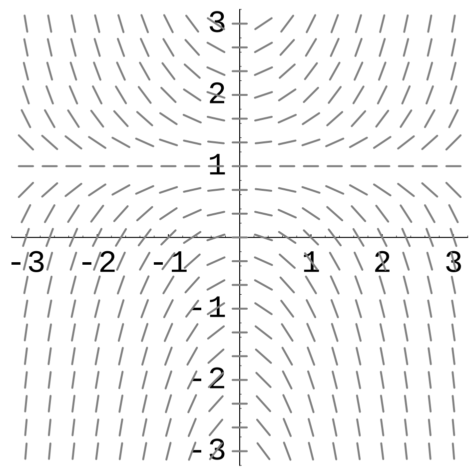
1. $\frac{dy}{dt} = ty + t$ 2. $\frac{dy}{dt} = y^2 + 1$ 3. $\frac{dy}{dt} = ty - t$ 4. $\frac{dy}{dt} = \sin t$

5. $\frac{dy}{dt} = y - t^2$ 6. $\frac{dy}{dt} = \cos t$ 7. $\frac{dy}{dt} = y + t^2$ 8. $\frac{dy}{dt} = 1 - y^2$

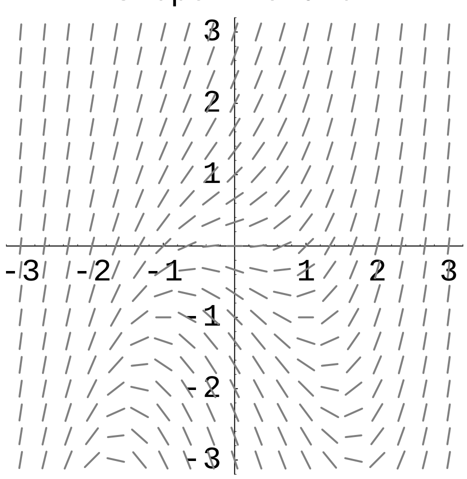
Slope Field A



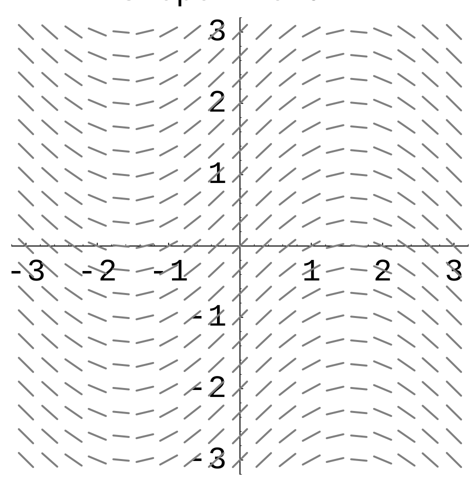
Slope Field B



Slope Field C



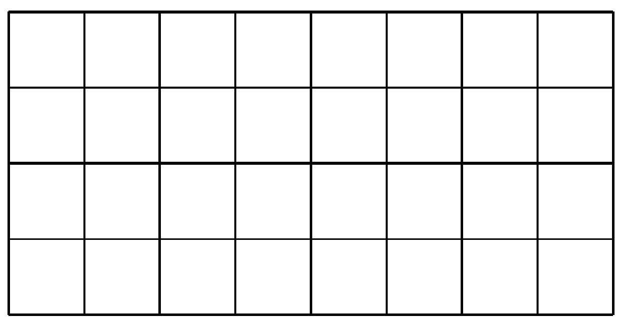
Slope Field D



2. (20 points) Consider the initial-value problem

$$\frac{dy}{dt} = 3 - y, \quad y(0) = 2.$$

(a) Draw the graph of the approximate solution that is obtained using Euler's method on the interval $[0, 1]$ with 4 subdivisions. Make sure that you show all of your calculations. Also, make sure that you label the axes on your graph and clearly indicate the scale on each axis. You may use a calculator and do all calculations to 2 decimal places if you wish.



(b) How does the graph of this approximate solution relate to the graph of the actual solution? Why?

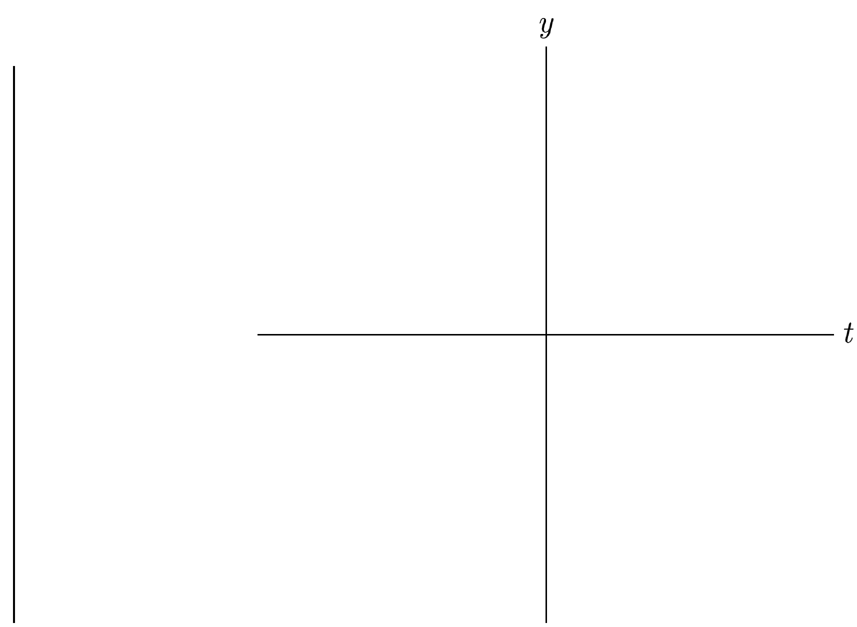
3. (22 points) Find the solution to the initial-value problem

$$\frac{dy}{dt} = \frac{2y}{t} + 2t^2, \quad y(-2) = 4.$$

4. (20 points) Consider the autonomous differential equation

$$\frac{dy}{dt} = y^4 - 2y^2.$$

On the left below, sketch the phase line for this equation and classify the equilibria (e.g., sink, ...). Then give a rough sketch of the graphs of the solutions to the equation on the ty-plane to the right of the phase line. **You will not get any credit unless you show calculations that justify your answer.**



5. (12 points) The U.S. population data that we used during the first week of class is given in the table to the right.

Year	U.S. Pop.
1790	3.9
1800	5.3
1810	7.2
1820	9.6
1830	12
1840	17
1850	23
1860	31
1870	38
1880	50
1890	62
1900	75
1910	91
1920	130
1930	122
1940	131
1950	151
1960	179
1970	203
1980	226
1990	249
2000	281

a. Estimate the relative growth rates of the population at the Years 1810 and 1990.

b. Using just the two numbers you obtained in part a, determine a logistic population model for this population (including the values of the parameters).

c. Using the model you derived in part b, estimate the carrying capacity.

MA 226 Exam 2A March 26, 2003

Name: ______________________ Last five digits of ID number: __________

Discussion Section (circle yours): M 12–1 M 2–3 M 3–4 T 2:30–3:30 T 3:30–4:30

Directions: Please do all of your work in this exam booklet and make sure that you cross out any work that we should ignore when we grade. **Books and extra papers are not permitted.** If you have a question about a problem, please ask. Remember: answers that are written logically and clearly will receive higher scores. There are 5 questions on 6 pages (not counting this cover page). Please make sure that you have all 6 pages of questions.

Do not write in the following box:

PROBLEM	POSSIBLE	SCORE
Name, etc.	2	
1	21	
2	20	
3	18	
4	18	
5	21	
TOTAL	100	

1. (21 points) Consider the second-order equation

$$3\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 0.$$

(a) Convert the equation to a first-order system.

(b) Calculate its eigenvalues.

(c) Calculate the "natural period" of the system.

2. (20 points) Compute the general solution of the linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -2 & -2 \\ -2 & 1 \end{pmatrix} \mathbf{Y}.$$

3. (18 points) Consider the damped harmonic oscillator that is modeled by the equation

$$2\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 3y = 0.$$

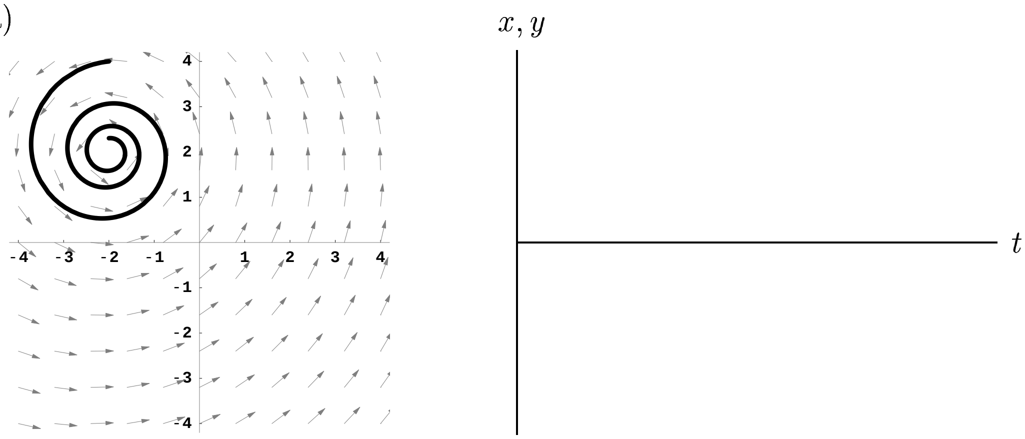
(a) Using a guessing technique, find two nonzero solutions $y_1(t)$ and $y_2(t)$ that are not multiples of one another. Make sure that you show the computations that go into your "guess."

(b) Convert this equation into a first-order system and determine the vector-valued solutions $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ that correspond to the solutions you obtained in part (a). (*Hint:* You do **not** need to calculate eigenvectors for this system.)

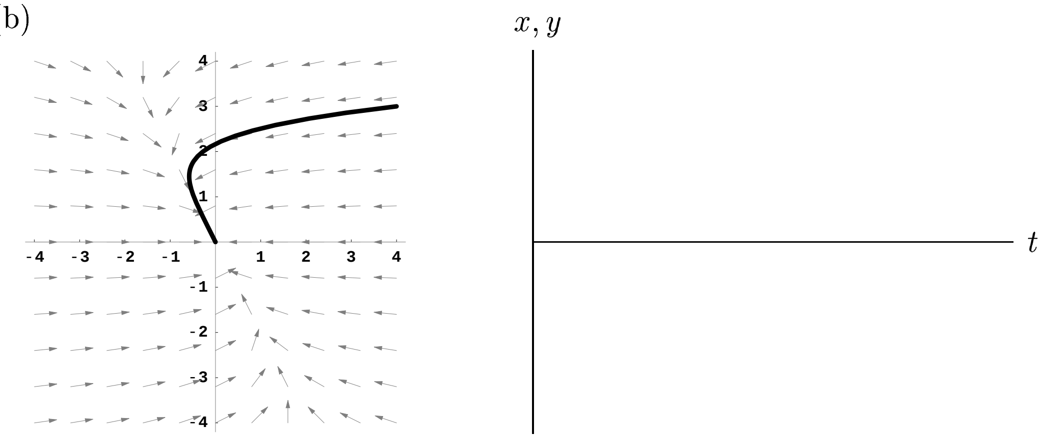
(c) In what way are $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ special solutions of the system? (One sentence answer.)

4. (18 points) Sketch the $x(t)$- and $y(t)$-graphs ($t \geq 0$) corresponding to the solution curves given below. Sketch the $x(t)$-graph using a solid curve and the $y(t)$-graph using a dashed curve, and put both curves on the same set of axes. No scale is necessary on the t-axis (the horizontal axis). On the vertical axis, use the same scale that is used on the vertical axis in the phase plane.

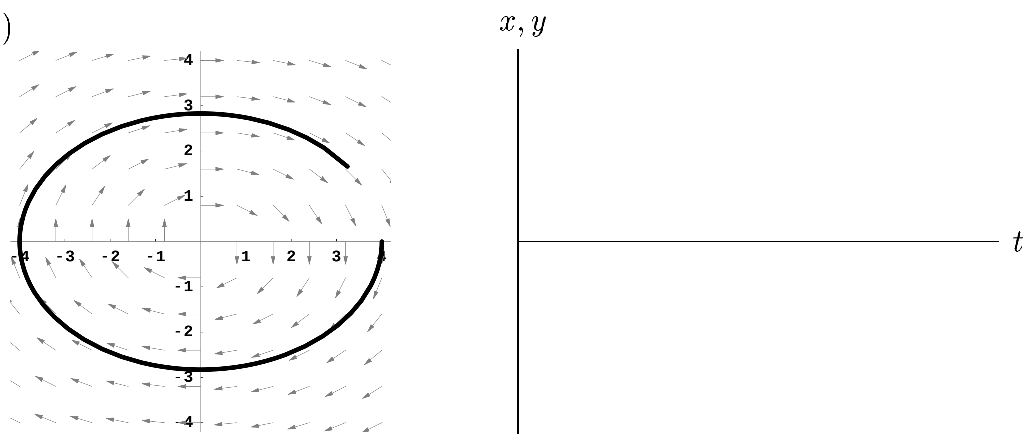
(a)



(b)



(c)



5. (21 points) On the next page, there are six matrices that can be used to form linear systems. There are also $x(t)$- and $y(t)$-graphs of three solutions to those systems. Pair each solution with its corresponding matrix, and provide a brief justification for your choice. **You will not receive any credit unless you provide a valid justification.**

   (i) The matrix for solution 1 is ____. My reason for choosing this answer is:

   (ii) The matrix for solution 2 is ____. My reason for choosing this answer is:

   (iii) The matrix for solution 3 is ____. My reason for choosing this answer is:

5. (continued) **Answer this question on the previous page.**

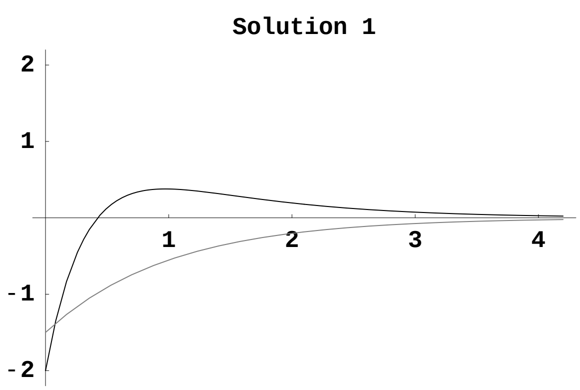
The six matrices:

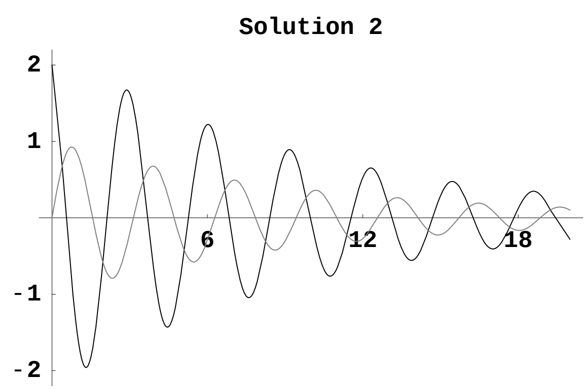
$$\mathbf{A} = \begin{pmatrix} -3 & -2 \\ 0 & -1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}$$

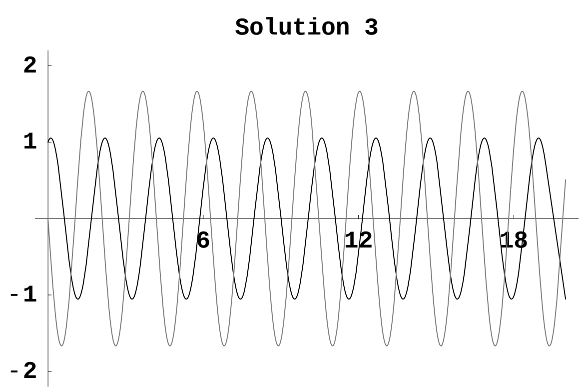
$$\mathbf{C} = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \qquad \mathbf{D} = \begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix}$$

$$\mathbf{E} = \begin{pmatrix} -1.1 & -2 \\ 2 & -1.1 \end{pmatrix} \qquad \mathbf{F} = \begin{pmatrix} -1.1 & -5 \\ 1 & 0.9 \end{pmatrix}$$

The $x(t)$- and $y(t)$-graphs of three solutions:







MA 226	Exam 3A	April 30, 2003

Name: ______________________________ Last five digits of ID number: ____________

Directions: Please do all of your work in this exam booklet and make sure that you cross out any work that we should ignore when we grade. **Books and extra papers are not permitted.** If you have a question about a problem, please ask. Remember: answers that are written logically and clearly will receive higher scores. There are 5 questions on 5 pages (not counting this cover page). Please make sure that you have all 5 pages of questions.

Do not write in the following box:

PROBLEM	POSSIBLE	SCORE
Name, etc.	2	
1	22	
2	22	
3	14	
4	22	
5	18	
TOTAL	100	

1. (22 points) Solve the initial-value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 0, \quad y(0) = 3, \quad y'(0) = -1.$$

2. (22 points) Consider the second-order equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 2\cos 3t.$$

(a) Determine a particular solution to this differential equation.

(b) Find the general solution to this differential equation.

3. (14 points) Compute $\mathcal{L}[u_2(t)\, t]$ **directly from the definition**.

4. (22 points) Solve the initial-value problem

$$\frac{dy}{dt} + 2y = 5u_3(t)\sin(t-3), \quad y(0) = 4.$$

5. (18 points) Consider the solution to the initial-value problem

$$\frac{d^2y}{dt^2} + 6y = \cos 2t, \quad y(0) = y'(0) = 0.$$

(a) Determine the angular frequency of the beats. **Include a one-sentence justification of your answer.**

(b) Determine the angular frequency of the rapid oscillations. **Include a one-sentence justification of your answer.**

(c) Use the information from parts (a) and (b) to give a rough sketch of the graph of the solution. **Make sure that you indicate a scale on both axes.**

NAME: ______________________________

Directions: Please do all of your work in this exam booklet and cross out any work that the grader should ignore. **Books and extra papers are not permitted.** If you have a question about a problem, please ask. Remember: Answers that are written logically and clearly will receive higher scores. There are 7 questions on 10 pages (not counting this cover page). Please make sure that you have all 10 pages of questions.

PROBLEM	POSSIBLE	SCORE
1	12	
2	14	
3	16	
4	12	
5	16	
6	14	
7	16	
TOTAL	100	

1. (12 points) Solve the initial-value problem

$$\frac{dy}{dt} = ty^2 + 3y^2, \quad y(0) = -1.$$

2. (14 points) Consider the system

$$\frac{dx}{dt} = 10 - x^2 - y^2$$
$$\frac{dy}{dt} = 3x - y.$$

(a) Determine all equilibrium points of this system.

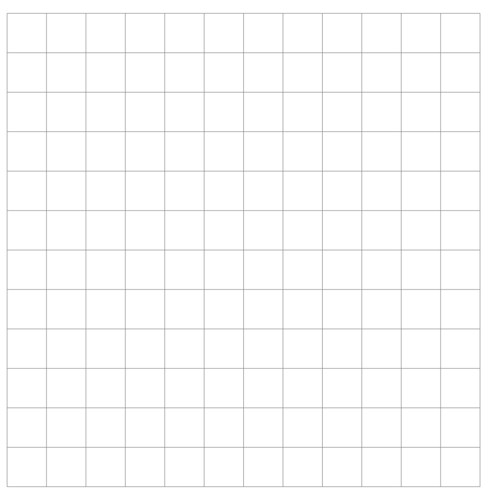
(b) Identify the types of the equilibrium points that you found in part a. In other words, determine if they are sinks, saddles, sources, ... Make sure that you indicate how you derived your answer.

3. (16 points) **Note that part c of this problem is on the next page.** Consider the linear system

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -4x - 4y.$$

(a) Determine the type of the equilibrium point at the origin and find all straight-line solutions. Make sure that you show the computations that justify your answers.

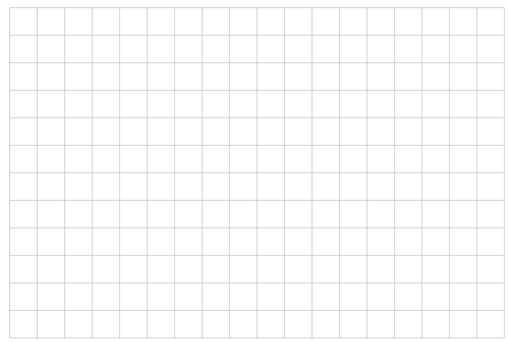
(b) Sketch the phase portrait for this system over the square $-6 \leq x \leq 6$ and $-6 \leq y \leq 6$.



3. (continued)

(c) Plot the initial conditions $A = (0, 4)$, $B = (4, 1)$, $C = (2, -4)$, and $D = (-5, 0)$ on your phase portrait on the previous page. Then plot the $x(t)$- and $y(t)$-graphs for $t \geq 0$ for the initial conditions A, B, C, and D below. Make sure that you label the axes, indicate a scale on the vertical axis, and distinguish the $x(t)$- from the $y(t)$-graph.

$x(t)$- and $y(t)$-graphs for the initial condition A



$x(t)$- and $y(t)$-graphs for the initial condition B

$x(t)$- and $y(t)$-graphs for the initial condition C

$x(t)$- and $y(t)$-graphs for the initial condition D

4. (12 points)

   (a) Find the general solution for the equation

$$\frac{dy}{dt} = -2y + 5\cos 3t.$$

   (b) Give a one-sentence justification of the fact that all solutions tend toward a steady-state solution as $t \to \infty$. Calculate the amplitude of the steady-state solution.

5. (16 points) **Note that parts c and d of this problem are on the next page.** Consider the initial-value problem

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 9y = 20u_2(t)\sin(t-2), \quad y(0) = 1, \quad y'(0) = 2.$$

(a) Describe the long-term behavior of the solution. *You need not do a lot of computation to answer this part of the problem, but you must provide a brief justification of your answer.*

(b) Calculate $\mathcal{L}^{-1}\left[\frac{s+6}{s^2+4s+9}\right]$

5. (continued)

(c) Calculate $\mathcal{L}^{-1}\left[e^{-2s}\left(\frac{s+2}{s^2+4s+9}-\frac{s-2}{s^2+1}\right)\right]$

(d) Calculate the Laplace transform $\mathcal{L}[y]$ for the solution $y(t)$ to the initial-value problem stated at the beginning of this problem. **You do not need to calculate a formula for $y(t)$.**

6. (14 points) A tank containing chocolate milk initially contains a mixture of 460 gallons of milk and 40 gallons of chocolate syrup. Milk is added to the tank at the rate of 8 gallons per minute and syrup is added to the tank at the rate of 2 gallons per minute. At the same time, chocolate milk is withdrawn at the rate of 10 gallons per minute. Assuming perfect mixing of milk and syrup:

   (a) Write an initial-value problem that describes the amount of syrup in the tank.

   (b) Sketch the phase line for the initial-value problem in part a, and determine how much syrup will be in the tank over the long term.

   (c) Determine how much syrup will be in the tank after 10 minutes.

7. (16 points) On the next page, there are 8 second-order linear equations and four graphs of solutions. Match each solution with its corresponding equation. Provide a brief justification for your choice. **You will not receive any credit unless you provide a valid justification.**

   (a) The equation for graph A is ____. My reason for choosing this answer is:

   (b) The equation for graph B is ____. My reason for choosing this answer is:

   (c) The equation for graph C is ____. My reason for choosing this answer is:

   (d) The equation for graph D is ____. My reason for choosing this answer is:

7. (continued) **Answer this question on the previous page.**

The 8 second-order linear equations:

1. $\dfrac{d^2y}{dt^2} + 16y = 0$

2. $\dfrac{d^2y}{dt^2} + 5\dfrac{dy}{dt} + y = 5\cos 2t$

3. $\dfrac{d^2y}{dt^2} + 5\dfrac{dy}{dt} + y = 5\cos 4t$

4. $2\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} + 10y = 0$

5. $2\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + 10y = 0$

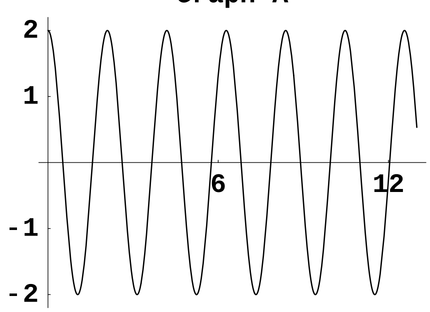
6. $\dfrac{d^2y}{dt^2} + 3y = \cos 11t$

7. $\dfrac{d^2y}{dt^2} + 9y = 0$

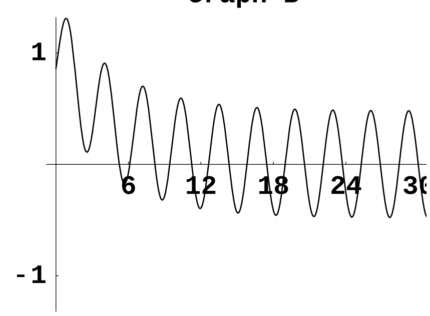
8. $\dfrac{d^2y}{dt^2} + 11y = \cos 3t$

The four graphs of solutions:

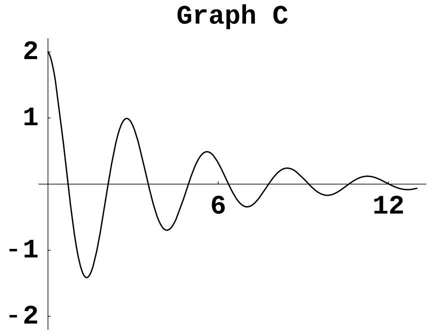
Graph A



Graph B



Graph C



Graph D