MA 226 Sample Examinations

Attached are the four examinations that were given in my MA 226 class in the Spring of 2004. You should use them to get an idea of the format of a typical test and to see the types of questions I ask. *You should not assume that the test questions this semester will be on the same topics.* In fact, you are always responsible for *all* of the material that we cover in class as well as *all* of the designated material from your text, and the best way to study for my exams is to be sure that you are very comfortable with the homework assignments and the examples that I present in class. My tests often vary in difficulty (as you can see from the attached), and your grade for the examination will be determined by a curve that will be announced in class after the examination is graded.

MA 226	Exam 1A	February 11, 2004

Name: ______________________ Last five digits of ID number: ____________

Discussion Section (circle yours): M 12–1 M 2–3 M 3–4 T 2:30–3:30 T 3:30–4:30

Directions: Please do all of your work in this exam booklet and make sure that you cross out any work that we should ignore when we grade. **Books and extra papers are not permitted.** If you have a question about a problem, please ask. Remember: answers that are written logically and clearly will receive higher scores. There are 5 questions on 5 pages (not counting this cover page). Please make sure that you have all 5 pages of questions.

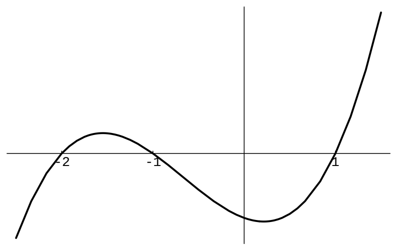
Do not write in the following box:

PROBLEM	POSSIBLE	SCORE
Name, etc.	2	
1	20	
2	18	
3	20	
4	20	
5	20	
TOTAL	100	

1. (20 points) Find the solution to the initial-value problem

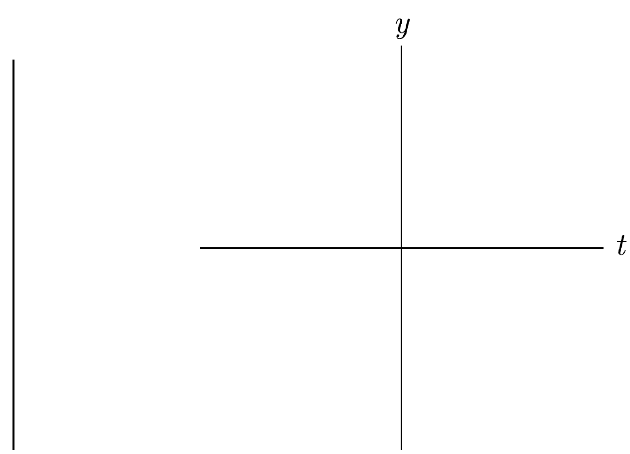
$$\frac{dy}{dt} = \frac{y}{t+1} + 4t + 4t^2, \quad y(1) = 10.$$

2. (18 points) Consider the differential equation $dy/dt = f(y)$ where $f(y)$ is given by the following graph:



(a) Give a rough sketch of the slope field that corresponds to this differential equation. (Include a scale on both axes.)

(b) On the left below, sketch of the phase line for this equation and classify the equilibria (e.g., sink, ...). Then on the right give a rough sketch of the graph of the solution that satisfies the initial condition $y(0) = 0$. (Include scales on all axes.)



3. (20 points) Consider the differential equation

$$\frac{dy}{dt} = -2t^3y^2.$$

(a) Find the general solution.

(b) Find the values of y_0 such that the solution with initial condition $y(-1) = y_0$ does not blow up (or down) in finite time. (Make sure that you show enough work to justify your answer.)

4. (20 points) Consider the differential equation

$$\frac{dy}{dt} = y^2 - 2(t+1)y + (t+1)^2.$$

(a) Show that the functions $y_1(t) = t$ and $y_2(t) = t+2$ are solutions of this differential equation. (Hint: Do not waste your time trying to come up with the general solution.)

(b) Using the Existence and Uniqueness Theorem, what can you say about $y_3(1)$ if $y_3(t)$ is the solution that satisfies the initial condition $y_3(0) = 1$? (Include a one-sentence justification for your answer.)

5. (20 points) A vat containing cherry cola initially contains a mixture of 450 gallons of cola and 50 gallons of cherry syrup. Cola is added to the vat at the rate of 7 gallons per minute, and cherry syrup is added to the tank at the rate of 3 gallons per minute. At the same time, cherry cola is withdrawn at the rate of 10 gallons per minute. Assuming that the cola and syrup are well mixed:

   (a) Write an initial-value problem that describes the amount of syrup in the tank.

   (b) Sketch the phase line for the initial-value problem in part (a), and determine how much syrup will be in the tank over the long term.

   (c) How much syrup will be in the tank after 10 minutes?

MA 226	Exam 2A	March 24, 2004

Name: ________________________________ Last five digits of ID number: __________

Discussion Section (circle yours): M 12–1 M 2–3 M 3–4 T 2:30–3:30 T 3:30–4:30

Directions: Please do all of your work in this exam booklet and make sure that you cross out any work that we should ignore when we grade. **Books and extra papers are not permitted.** If you have a question about a problem, please ask. Remember: answers that are written logically and clearly will receive higher scores. There are 5 questions on 6 pages (not counting this cover page). Please make sure that you have all 6 pages of questions.

Do not write in the following box:

PROBLEM	POSSIBLE	SCORE
Name, etc.	3	
1	20	
2	21	
3	20	
4	20	
5	16	
TOTAL	100	

1. (20 points) Consider the following 8 first-order systems:

1. $\dfrac{dx}{dt} = x$, $\dfrac{dy}{dt} = x + y$
2. $\dfrac{dx}{dt} = x + y$, $\dfrac{dy}{dt} = y$
3. $\dfrac{dx}{dt} = -x$, $\dfrac{dy}{dt} = y^2 - 1$
4. $\dfrac{dx}{dt} = x$, $\dfrac{dy}{dt} = y^2 - 1$
5. $\dfrac{dx}{dt} = -2x$, $\dfrac{dy}{dt} = -y$
6. $\dfrac{dx}{dt} = -x$, $\dfrac{dy}{dt} = -2y$
7. $\dfrac{dx}{dt} = x + 1$, $\dfrac{dy}{dt} = y^3$
8. $\dfrac{dx}{dt} = x + 1$, $\dfrac{dy}{dt} = y^2$

Four of the associated direction fields are shown on the next page. Pair the direction fields with their corresponding systems. Provide a brief justification for your choice. **You will not receive any credit for your answer unless you provide a valid justification.**

A. The system for direction field A is ____. My reason for choosing this answer is:

B. The system for direction field B is ____. My reason for choosing this answer is:

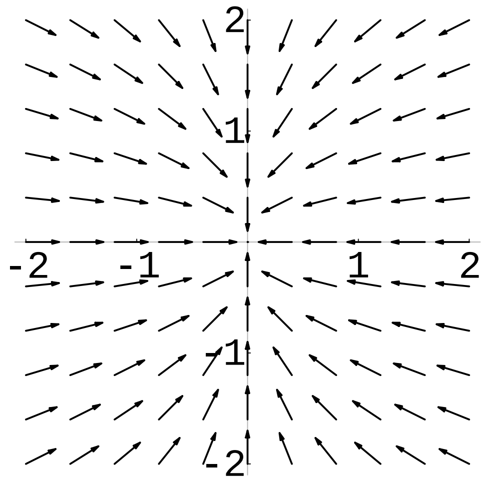
C. The system for direction field C is ____. My reason for choosing this answer is:

D. The system for direction field D is ____. My reason for choosing this answer is:

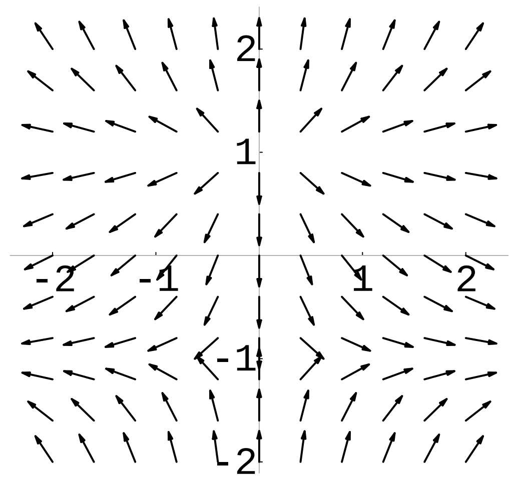
1. (continued) **Answer this question on the previous page.** The systems are provided here solely for your convenience:

1. $\dfrac{dx}{dt} = x$, $\dfrac{dy}{dt} = x + y$
2. $\dfrac{dx}{dt} = x + y$, $\dfrac{dy}{dt} = y$
3. $\dfrac{dx}{dt} = -x$, $\dfrac{dy}{dt} = y^2 - 1$
4. $\dfrac{dx}{dt} = x$, $\dfrac{dy}{dt} = y^2 - 1$
5. $\dfrac{dx}{dt} = -2x$, $\dfrac{dy}{dt} = -y$
6. $\dfrac{dx}{dt} = -x$, $\dfrac{dy}{dt} = -2y$
7. $\dfrac{dx}{dt} = x + 1$, $\dfrac{dy}{dt} = y^3$
8. $\dfrac{dx}{dt} = x + 1$, $\dfrac{dy}{dt} = y^2$

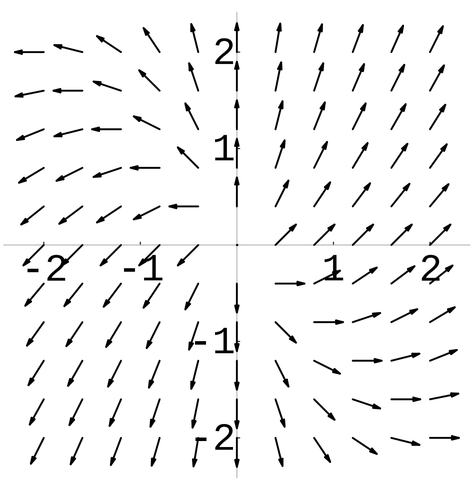
Direction Field A



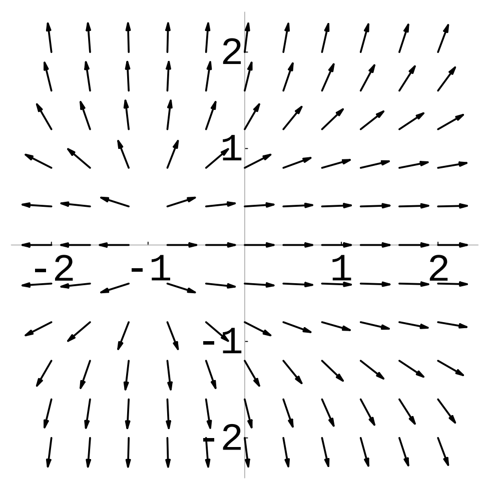
Direction Field B



Direction Field C



Direction Field D



2. (21 points)

   (a) Convert the second-order equation $\frac{d^2y}{dt^2} + b\frac{dy}{dt} + 7y = 0$ into a first-order system.

   (b) For what values of b is the equilibrium point at the origin a spiral sink in the system in part a?

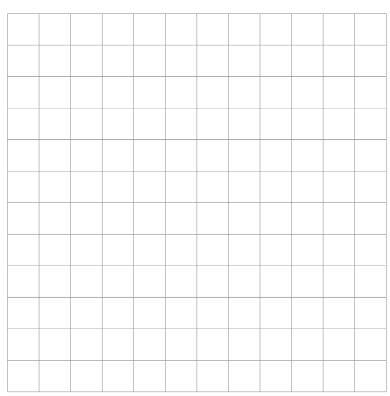
   (c) Compute the equilibrium point(s) for the equation $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 7y + 2h(y) = 10$, where $h(y) = y$ if $y \geq 0$ and $h(y) = 0$ if $y \leq 0$. Make sure that you consider all cases in your calculation.

3. (20 points) Consider the linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \mathbf{Y}.$$

(a) Determine the type of the equilibrium point at the origin and find *all* straight-line solutions. Make sure that you show the computations that justify your answers.

(b) Sketch the phase portrait of this system over the square $-6 \leq x \leq 6$ and $-6 \leq y \leq 6$.



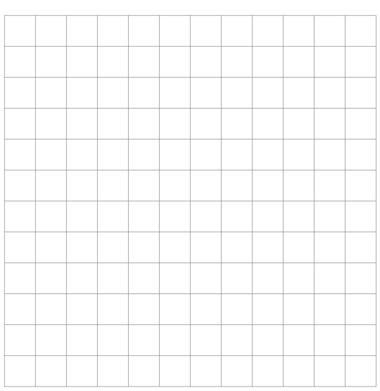
4. (20 points) Consider the initial-value problem

$$\frac{dx}{dt} = x + y \qquad \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$
$$\frac{dy}{dt} = 2x - y,$$

(a) Using Euler's method with 4 steps, calculate an approximate solution over the interval $0 \leq t \leq 1$, and enter the results in the table provided. Make sure that you show enough calculations so that the grader can understand how you obtained your answer. You may use a calculator and do all calculations to two decimal places if you wish.

k	x_k	y_k
0	0	2
1		
2		
3		
4		

(b) On the left-hand grid, sketch the approximate solution curve in the xy-phase plane, and on the right-hand grid, sketch the graphs of the approximations to the functions $x(t)$ and $y(t)$. Make sure that you put labels and scales on all axes.


solution curve in xy-plane


$x(t)$- and $y(t)$-graphs

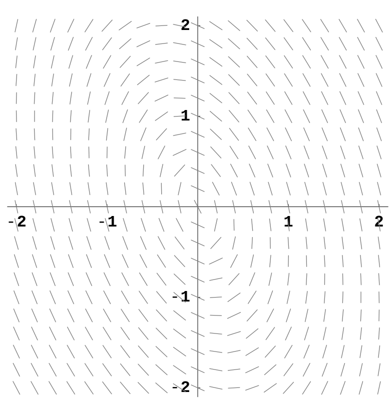
5. (16 points) The linear system

$$\frac{dx}{dt} = 1.1x + 2y$$
$$\frac{dy}{dt} = -5x - 0.9y$$

has complex eigenvalues.

(a) Compute these eigenvalues. What type of equilibrium point is the origin? What can you say about the solutions to the system from these eigenvalues alone? (Be as specific as possible.)

(b) The slope field for this system is shown below. Using this field, give a rough sketch of the solution curve whose initial condition is $(1, 0)$. Sketch the curve for positive time only ($t \geq 0$) and give a brief justification of why the curve looks the way it does.



MA 226	Exam 3A	April 28, 2004

Name: ______________________ Last five digits of ID number: ____________

Discussion Section (circle yours): M 12–1 M 2–3 M 3–4 T 2:30–3:30 T 3:30–4:30

Directions: Please do all of your work in this exam booklet and make sure that you cross out any work that we should ignore when we grade. **Books and extra papers are not permitted.** If you have a question about a problem, please ask. Remember: answers that are written logically and clearly will receive higher scores. There are 5 questions on 5 pages (not counting this cover page). Please make sure that you have all 5 pages of questions.

Do not write in the following box:

PROBLEM	POSSIBLE	SCORE
Name, etc.	3	
1	17	
2	20	
3	20	
4	20	
5	20	
TOTAL	100	

1. (17 points) Solve the initial-value problem

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 0, \quad y(0) = 2, \quad y'(0) = -10.$$

2. (20 points) Consider the system

$$\frac{dx}{dt} = x - 3y^2$$
$$\frac{dy}{dt} = x - 3y - 6.$$

(a) Determine all equilibrium points of this system.

(b) Identify the types of the equilibrium points that you found in Part a. In other words, determine if they are sinks, saddles, sources, ... Make sure that you indicate how you derived your answer.

3. (20 points) Consider the second-order equation

$$\frac{d^2y}{dt^2} + 4y = 3\cos 2t.$$

(a) Determine a particular solution to this differential equation.

(b) Find the general solution to this differential equation.

4. (20 points)

(a) Solve the initial-value problem

$$\frac{d^2y}{dt^2} + \omega^2 y = 0, \quad y(0) = 0, \quad y'(0) = \omega$$

without using the Laplace transform.

(b) Apply the Laplace transform to both sides of the initial-value problem in part a and calculate $\mathcal{L}[y]$.

(c) What can you conclude from these two calculations?

5. (20 points)

(a) Calculate $\mathcal{L}^{-1}\left[\dfrac{2s}{s^2-6s+11}\right]$.

(b) Compute $\mathcal{L}[u_3(t)\, e^{t-3}]$ **directly from the definition of the Laplace transform**.

MA 226	Final A	May 7, 2004

Name: ______________________ Last five digits of ID number: __________

Directions: Please do all of your work in this exam booklet and make sure that you cross out any work that we should ignore when we grade. **Books and extra papers are not permitted.** If you have a question about a problem, please ask. Remember: answers that are written logically and clearly will receive higher scores. There are 7 questions on 9 pages (not counting this cover page). Please make sure that you have all 9 pages of questions.

Do not write in the following box:

PROBLEM	POSSIBLE	SCORE
1	14	
2	14	
3	14	
4	14	
5	14	
6	15	
7	15	
TOTAL	100	

1. (14 points) Solve the initial-value problem

$$\frac{dy}{dt} = -\frac{y}{t+1} + 2, \quad y(0) = 3.$$

2. (14 points) Solve the initial-value problem

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = 0, \quad y(0) = 2, \quad y'(0) = -8.$$

3. (14 points) Solve the initial-value problem

$$\frac{dy}{dt} - 4y = 50u_2(t)\sin\left(3(t-2)\right), \quad y(0) = 5.$$

4. (14 points) **Note that part b of this problem is on the next page.**

(a) Compute a periodic solution for the second-order equation

$$\frac{d^2y}{dt^2} + \frac{1}{10}\frac{dy}{dt} + 13y + 4h(y) = 10 + \frac{1}{10}\sin 4t,$$

where $h(y) = y$ if $y \geq 0$ and $h(y) = 0$ if $y \leq 0$.

4. (continued)

(b) In a brief paragraph, compare and contrast the behavior of the solutions to the differential equation

$$\frac{d^2y}{dt^2} + \frac{1}{10}\frac{dy}{dt} + 17y = 10 + \frac{1}{10}\sin 4t$$

with the solutions of the equation given in part a on the previous page.

5. (14 points) Consider the one-parameter family of equations

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + 3y = 0,$$

where the parameter b satisfies $-\infty < b < \infty$.

(a) Convert the family into a one-parameter family of first-order systems.

(b) Draw the curve in the trace-determinant plane obtained by varying the parameter.

(c) Determine all bifurcation values and briefly discuss the different types of behavior exhibited by this one-parameter family.

6. (15 points) On the next page, there are six matrices that can be used to form linear systems. There are also $x(t)$- and $y(t)$-graphs of three solutions to those systems. Pair each solution with its corresponding matrix, and provide a brief justification for your choice. **You will not receive any credit unless you provide a valid justification.**

   (i) The matrix for solution 1 is ____. My reason for choosing this answer is:

   (ii) The matrix for solution 2 is ____. My reason for choosing this answer is:

   (iii) The matrix for solution 3 is ____. My reason for choosing this answer is:

6. (continued) **Answer this question on the previous page.**

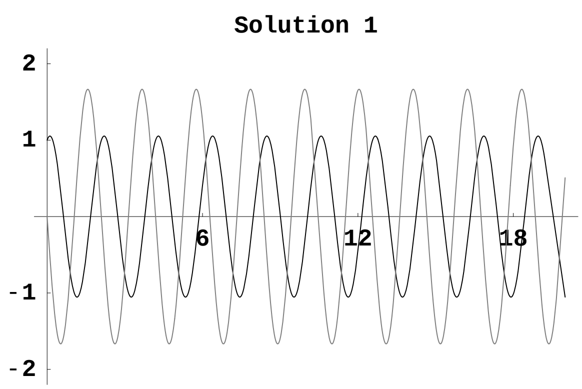
The six matrices:

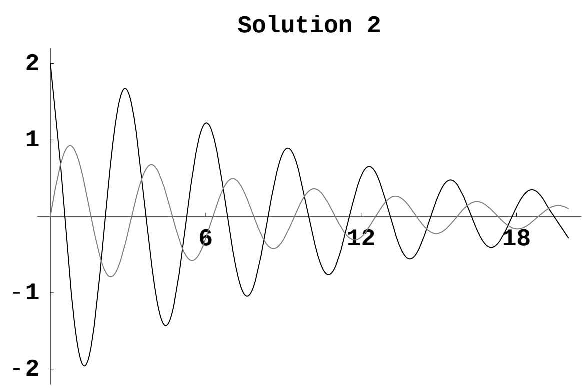
$$\mathbf{A} = \begin{pmatrix} -3 & -2 \\ 0 & -1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}$$

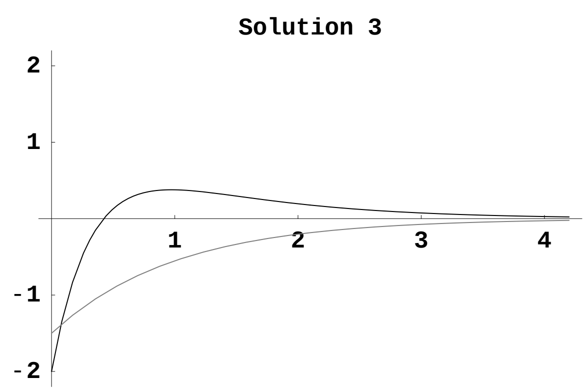
$$\mathbf{C} = \begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix} \qquad \mathbf{D} = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix}$$

$$\mathbf{E} = \begin{pmatrix} -1.1 & -5 \\ 1 & 0.9 \end{pmatrix} \qquad \mathbf{F} = \begin{pmatrix} -1.1 & -2 \\ 2 & -1.1 \end{pmatrix}$$

The $x(t)$- and $y(t)$-graphs of three solutions:







7. (15 points) A simple model of weight gain/loss assumes that an adult male needs to consume 20 calories per day per pound to maintain his weight. If he consumes more or fewer calories, then his weight changes in proportion to the difference between the number of calories consumed and the number of calories needed to maintain his current weight. Suppose Sebastian starts the year 2004 weighting 160 pounds, and he consumes 3000 calories each day.

   (a) Write an initial-value problem that describes Sebastian's weight. (Use $k > 0$ to represent the proportionality constant.)

   (b) Sketch the phase line for the initial-value problem in part (a), and determine Sebastian's weight over the long term.

   (c) Suppose that the proportionality constant $k = 0.0005$. How much does Sebastian weigh after 100 days?