MA 226 Spring 2016
Project 2

Due: At the **start of class** on Wednesday, April 27.

Group project: This is a group project with each group having either three or four members. Once a group begins work on the project, its membership cannot change. Consequently establishing your group must be your first step in this project. Each group will submit one report, and all members of the group will receive the same grade for this project.

Project description: This project is intended to help you become familiar with the long-term behavior of solutions to certain forced second-order equations and with the relationship between the amplitude of the forcing and the periodicity of solutions to a certain nonlinear equation.

Using all of the methods that we have developed in this course, you should analyze and compare the long-term behavior of the solutions to two forced second-order equations.

The first equation is the forced harmonic oscillator

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + k_0 x = \cos \omega t.$$

As usual, b is the damping coefficient, k_0 is the spring constant (based on your ID numbers), and ω is the angular frequency of the forcing.

1. Undamped and forced linear equation: Consider the case where $b = 0$ and determine a particular solution to the equation. Then using either a numerical solver or formulae for the solutions, estimate the amplitudes of the solutions for frequencies ω in the interval $0 \leq \omega \leq 3$. How do the amplitudes in the nonresonant cases relate to the amplitude in the resonant case?

2. Damped and forced linear equation: Assume that damping is present, that is, $b > 0$, along with the forcing. How long does it take any given solution to get close to the steady-state solution? Using either a numerical solver or formulae for the solutions, estimate the maximum amplitudes of the solutions for frequencies ω in the interval $0 \leq \omega \leq 3$ for various values of b. Graph the maximum amplitude as a function of frequency for different values of b, all on the same set of axes. What happens to these graphs as $b \to 0$?

The second equation is a forced version of one of the nonlinear equations (equation 4) that we studied in the first project. The equation is

$$\frac{d^2x}{dt^2} + b_0(x^2 - 1)\frac{dx}{dt} + x = \epsilon\, b_0 \sin t$$

where b_0 is the parameter determined by your ID numbers (see below).

3. Search for periodic solutions to this forced equation for values of ϵ in the interval $0.01 < \epsilon < 0.66$. For each periodic solution that you find, specify the value of the parameter ϵ, the initial condition, and the period. (See the comment below regarding the fact that periodic solutions come in groups.)

Parameter values: There are two parameter values k_0 and b_0 that are determined by the last (single) digits of your BU ID numbers. The parameter k_0 is used with the damped forced harmonic oscillator and the parameter b_0 is used with the forced van der Pol equation. The values of these two parameters are the same, but I have used different notation to emphasize the fact that their meanings are dramatically different. Use $k_0 = 5 + 0.1a$ where a is the average (accurate to two decimal places) of the last digits of all members in the group. For example, if the last digits are 0, 1, 2, and 2, then $a = 1.25$ and $k_0 = 5.12$. Likewise, $b_0 = 5.12$.

Some history: The nonlinear equation is called the forced van der Pol equation. Van der Pol studied this equation as a model for a vacuum tube triode circuit approximately 90 years ago. He and his colleague, van der Mark, noted the existence of two stable periodic solutions with different periods for certain values of the parameter ϵ. Their work inspired two English mathematicians, Mary Cartwright and J. E. Littlewood, to study this equation in the 1940s and 50s, and they were able to prove that there are intervals of parameters for which there are two stable periodic solutions with different periods. These parameter values yield "chaotic" solutions. Since this work was roughly fifteen years prior to Lorenz's famous work on the Lorenz attractor (see Section 2.8 of our textbook), one could claim that chaos theory actually started with the work of Cartwright and Littlewood.

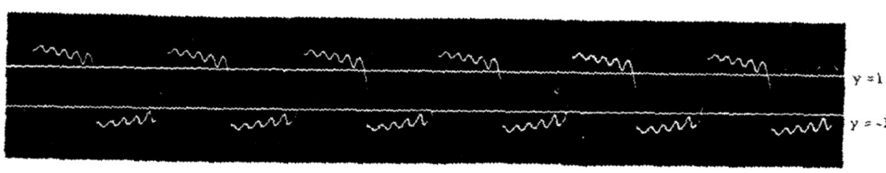
Even though the existence of these stable periodic solutions has been known for more than 65 years, there are relatively few graphs of these solutions in the literature (as far as I can tell). The next page shows two from a paper of Littlewood published in 1963.

While they are on C_2, Γ_1 and Γ_2 are walking a tight-rope. The unstable periodic Γ behave in this way just before each of their shoot-throughs, which are many when the period of Γ is long, and they slice or pull differently each time.

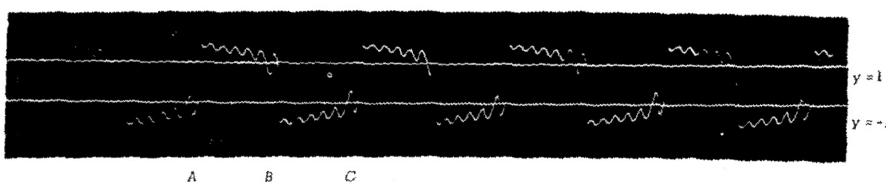
I have now to report a very surprising development, an electronic calculation discovery by A. Warren,[6] under the supervision of H. P. F. Swinnerton-Dyer. His k is 5, which is a good bet for being "large." For a certain $\underline{b}$ he finds two stable periods of orders 11 and 25, completely breaking the $2n \pm 1$ rule. Photographs of these are given in Fig. 3.

We can interpret the photographs in terms of the " k large" behavior we have been describing, at least up to a point. We must, of course, regard $\underline{b}$ as being in a transitional interval, and Γ_{11} as the stable Γ we said above might be expected. (Γ_{11} does not make any dips, but this is normal in part of the range of $\underline{b}$.)

The photograph of Γ_{25} reveals that it follows part of a C_2 at A, B, and C (pulling at B and invertedly slicing at A and C). The surprise, of course, is that, its three "exponentially" dangerous walks notwithstanding, it succeeds in being stable. After this there seems no reason why, for a single $\underline{b}$ and different initial conditions, slices and pulls could not happen in a variety of ways, in which case there could be 3 or more different stable periods.



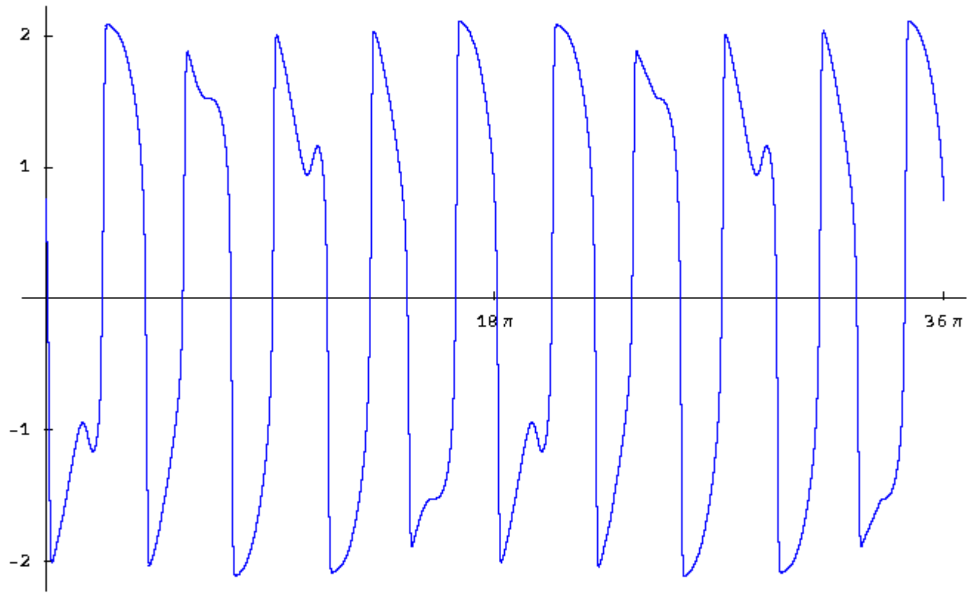
Stable trajectory - sub-harmonic order 11



Stable trajectory - sub-harmonic order 25

Figure 3

Here's one of period 18π that I computed.



Finding periodic solutions of a forced equation is more difficult than finding periodic solutions of autonomous equations. In the autonomous case, any solution curve that comes back to where it started in the phase plane corresponds to a periodic solution. Unfortunately, in the nonautonomous case, the vector field varies in time, and solution curves in the phase plane can cross.

It is a little tricky to use a computer to tell if you have found a periodic solution, but for this equation you can take advantage of the fact that the equation is periodic in t with period 2π. So if a solution goes for 2π units of time and ends up back at the same initial condition, then the solution has to be periodic. (Why? This point of view is discussed on pp. 538–549 of our text.) The same observation holds for any integer multiple of 2π. (Note that my solution has period 18π.) Also, if you find a periodic solution with a period of $2n\pi$ for $n > 1$, then you have actually found n different periodic solutions because you can translate any such periodic solution by any multiple of 2π and get another solution. (Why?) So my periodic solution above actually represents nine different periodic solutions.

`HPGSystemSolver` can help you with your search, but don't use the default step size. For this differential equation, I recommend that you set the step size to 0.001 (or even slightly smaller).

Poincaré map: For the forced van der Pol equation, you may want to use a new program called `PMapVDP` in conjunction with `HPGSystemSolver`. I have implemented an on-line version of the program using my department's *webMathematica* site. It works fine, but only two users can run it at the same time. My guess is that this will not be a problem until the night before the project is due. Here is the URL:

`http://locutus.bu.edu:8080/webMathematica/PBExamples/226project2-2016.jsp`

I have placed an active link to this program on the course web page just below this project statement.

Your report: Address the questions in each item in the project description in the form of a short essay. Reports that present the results in creative and efficient ways will receive higher grades.

Your report should be no longer than four typewritten pages, and it should address all of the questions in the project description. Your report should include a cover page followed by an academic conduct signature page. A blank signature page is provided at the end of this document. Neither of these two pages count as part of the four-page limit.

You may provide as many illustrations from the computer as you wish, but the relevance of each illustration to your report must be evident. Illustrations are part of the four-page limit. Please insert your illustrations at appropriate places in your report rather than attaching them to the end of the report.

Asking questions: Questions by email about the project are not allowed. Questions about the project will be answered using the discussion forum for our MA226 course on edge.edx.org. I will check the forum for questions once each day. The last time that I will check the forum will be noon on Monday, April 25. Do not wait until the last minute to ask questions.

Academic Conduct: Your work and conduct in this course are governed by the Boston University Academic Conduct Code. A copy of this code is available at

http://www.bu.edu/academics/files/2011/08/AcademicConductCode.pdf

This code is designed to promote high standards of academic honesty and integrity as well as fairness. It is your responsibility to know and follow the provisions of the code. **In particular, all work that you submit in this course must be your original work.** For example, you can only discuss your project with other members of your group or with Professor Blanchard. Moreover, the computations that you do for your report as well as the text of your report must be original to your group. All group members are responsible for all aspects of the report. If you have a question about any aspect of academic conduct, please ask.

We understand that this project report is governed by the Boston University Academic Conduct Code. In particular, this code requires that the work submitted here is our original work. We declare that all text and illustrations in this report were produced by us without the assistance of external sources and that all of the computations presented in this report were performed by us. Moreover, we have not performed any Google searches or otherwise used the internet to assist us with this report in any way that has not been explicitly approved by Professor Blanchard. We also affirm that we have not discussed our results with anyone other than Professor Blanchard. We understand that our group may be required to present our results orally to Professor Blanchard and Eric.

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