Method of integrating factors:

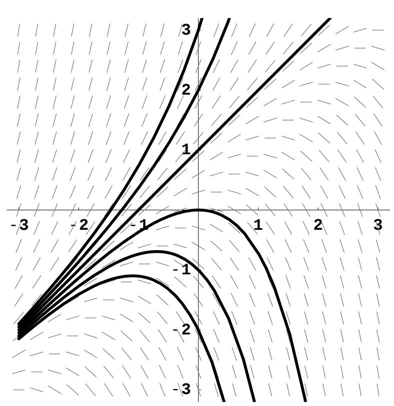
Summary from last class: Given a linear differential equation of the form

$$\frac{dy}{dt} + a(t)y = r(t),$$

the integrating factor (magic function) is

$$\mu(t) = e^{\left(\int a(t)\, dt\right)}.$$

Example. $\frac{dy}{dt} = y - t$



Linearity Principles

Why are linear equations so much more amenable to analytic techniques than nonlinear equations? The reason is that the solutions satisfy important linearity principles.

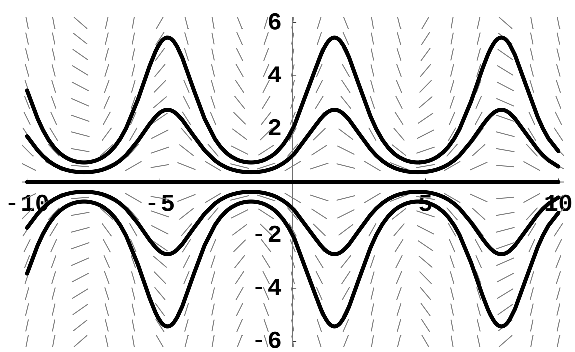
Let's begin with homogeneous linear equations:

Linearity Principle If $y_1(t)$ is a solution of a homogeneous linear differential equation

$$\frac{dy}{dt} + a(t)y = 0,$$

then any *constant* multiple $y_k(t) = ky_1(t)$ of $y_1(t)$ is also a solution. In other words, given a constant $k \neq 1$ and a solution $y_1(t)$, we obtain another solution by multiplying $y_1(t)$ by k.

Example. $\frac{dy}{dt} = (\cos t)y$



Note that the Linearity Principle is not true for nonlinear equations. For example, consider

$$\frac{dy}{dt} = y^2.$$

Check that one solution is

$$y_1(t) = \frac{1}{1-t},$$

and then check that

$$y_2(t) = 2y_1(t) = \frac{2}{1-t}$$

is not a solution.

There is a similar "linearity" principle for nonhomogeneous linear equations:

Extended Linearity Principle For First-Order Equations Consider a first-order, nonhomogeneous, linear equation

$$\frac{dy}{dt} + a(t)y = r(t)$$

and its associated homogeneous equation

$$\frac{dy}{dt} + a(t)y = 0.$$

1. If $y_h(t)$ is any solution of the homogeneous equation and $y_p(t)$ ("p" for particular solution) is *any* solution of the nonhomogeneous equation, then $y_h(t) + y_p(t)$ is also a solution of the nonhomogeneous equation.

2. Suppose $y_p(t)$ and $y_q(t)$ are two solutions of the nonhomogeneous equation. Then $y_p(t) - y_q(t)$ is a solution of the associated homogeneous equation.

Therefore, if $y_h(t)$ is nonzero, $ky_h(t) + y_p(t)$ is the general solution of the nonhomogeneous equation.

Example. Let's return to $\frac{dy}{dt} = y - t$.