

The Linearity Principle lets us produce many solutions from a few. Recall the example we did last class:

Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{Y}$$

and the two solutions

$$\mathbf{Y}_1(t) = \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}.$$

Any linear combination of $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ is also a solution to the system.

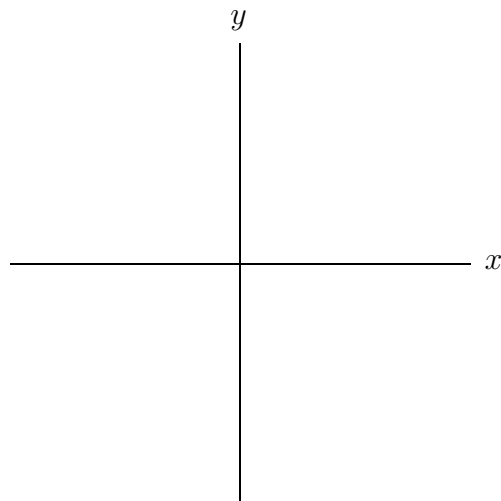
We can use this observation to solve any initial-value problem involving this system.

Example. Solve

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

How do we find a few solutions to get started?

Let's return to our example and see what is so special about $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$.



We want nonzero initial conditions \mathbf{Y}_0 (vectors) so that

$$\mathbf{A}\mathbf{Y}_0 = \lambda\mathbf{Y}_0$$

for some scalar λ .

Terminology: The scalar λ is called an *eigenvalue* of the matrix \mathbf{A} and the vector \mathbf{Y}_0 is called an *eigenvector* associated to the eigenvalue λ .

“Straight-line” Solutions. Suppose that

$$\mathbf{A}\mathbf{Y}_0 = \lambda\mathbf{Y}_0$$

for some nonzero vector \mathbf{Y}_0 and some scalar λ . Then the function

$$\mathbf{Y}(t) = e^{\lambda t}\mathbf{Y}_0$$

is a solution to the linear differential equation

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}.$$