Recall from last class that the eigenvalue/eigenvector relationship yields basic solutions to the linear system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}.$$

"Straight-line" Solutions. Suppose that

$$\mathbf{AY}_0 = \lambda \mathbf{Y}_0$$

for some nonzero vector $\mathbf{Y}_0$ and some scalar λ. Then the function

$$\mathbf{Y}(t) = e^{\lambda t}\mathbf{Y}_0$$

is a solution to the linear differential equation

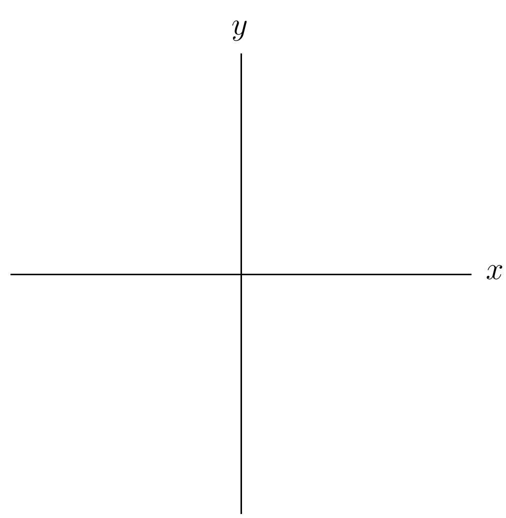
$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}.$$

How do we find eigenvalues and eigenvectors given the matrix $\mathbf{A}$?

Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 4 & -5 \\ -2 & 1 \end{pmatrix}\mathbf{Y}.$$

First let's see what `MatrixFields` tells us about the eigenvalues and eigenvectors of the matrix $\mathbf{A}$.



Aside from the theory of algebraic linear equations

For what matrices $\mathbf{B}$ does the equation $\mathbf{BY} = \mathbf{0}$ have nontrivial solutions?

Singular Matrices The matrix equation

$$\mathbf{BY} = \mathbf{0}$$

has nontrivial solutions $\mathbf{Y}$ if and only if

$$\det \mathbf{B} = 0.$$

Note: Most matrices are nonsingular.

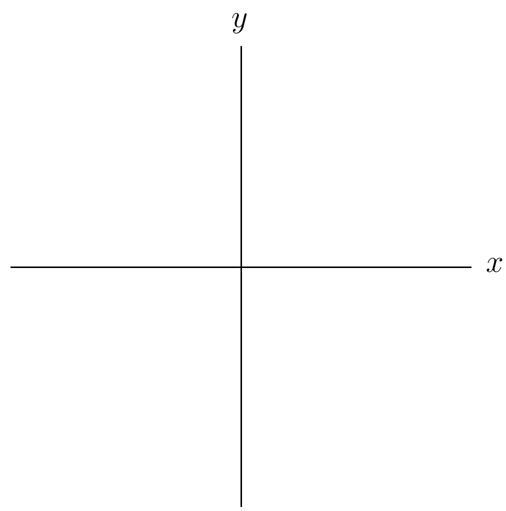
Now let's use this theorem to find eigenvalues and eigenvectors:

Back to the example:

Example. Find the general solution to

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 4 & -5 \\ -2 & 1 \end{pmatrix} \mathbf{Y}.$$

Using `HPGSystemSolver`, we plot the phase portrait for this system.



Facts about eigenvalues and eigenvectors: Given a 2×2 matrix $\mathbf{A}$,

1. The characteristic equation can have two real roots, one real root of multiplicity two, or two complex conjugate roots.

2. Given an eigenvector $\mathbf{Y}_0$ associated to an eigenvalue λ, then any nonzero scalar multiple $\mathbf{Y}_0$ is also an eigenvector associated to λ.

Summary of Case of Two Distinct Real Eigenvalues

Suppose $\mathbf{A}$ is a matrix with two eigenvalues λ_1 and λ_2. To be consistent, we'll assume that $\lambda_1 < \lambda_2$, that $\mathbf{V}_1$ is an eigenvector associated to λ_1, and that $\mathbf{V}_2$ is an eigenvector associated to λ_2. The general solution of

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

is

$$\mathbf{Y}(t) = k_1 e^{\lambda_1 t}\mathbf{V}_1 + k_2 e^{\lambda_2 t}\mathbf{V}_2.$$

Case 1: $\lambda_1 < \lambda_2 < 0$.