Last class we almost did a complete job of analyzing sinusoidal forcing without damping. Today we discuss one very important case—resonance.

Example.

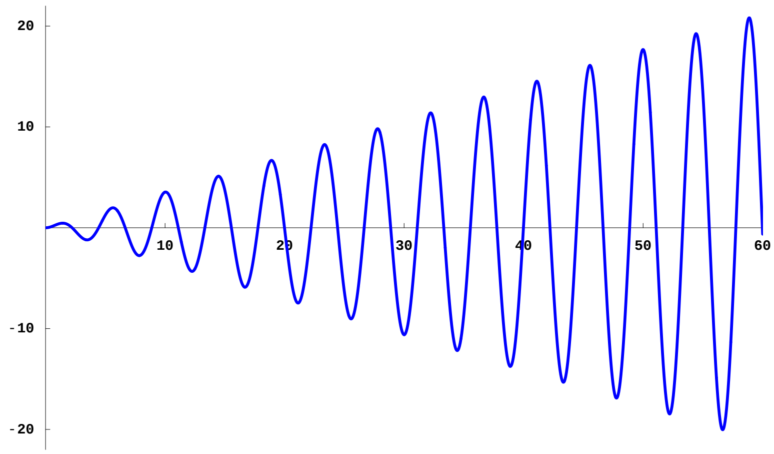
$$\frac{d^2y}{dt^2} + 2y = \cos\sqrt{2}\,t$$

The complexified equation is

$$\frac{d^2y}{dt^2} + 2y = e^{i\sqrt{2}\,t}.$$

What guess should we use?

Here is the graph of a typical solution:



Linearization:

We would like to apply what we know about linear systems to nonlinear systems.

Example. Consider the van der Pol equation

$$\frac{d^2x}{dt^2} + (x^2 - 1)\frac{dx}{dt} + x = 0.$$

The corresponding system is

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = (1 - x^2)y - x.$$

Let's calculate the equilibria:

The linearized system near $(0,0)$ is

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{Y}.$$

Example. Consider the (undamped) pendulum

$$\frac{d^2x}{dt^2} + \sin x = 0.$$

The corresponding system is

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -\sin x.$$

Let's calculate the equilibria:

The linearized system near $(0, 0)$ is

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{Y}.$$

The linearized system near $(\pi, 0)$ is

Given the (nonlinear) system

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y),$$

its **Jacobian** at the point (x_0, y_0) is the linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix} \mathbf{Y}.$$

For the pendulum, we obtain the two linearizations:

For the van der Pol equation, we obtain the linearization: