Linear systems

Last class we started to discuss systems that are linear. They are the ones that can be written in terms of matrix multiplication

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}.$$

Recall two examples that we have already discussed.

Example 1. We have already calculated the general solution to the partially decoupled system

$$\frac{dx}{dt} = 2y - x$$
$$\frac{dy}{dt} = y.$$

It is

$$x(t) = y_0 e^t + (x_0 - y_0)e^{-t}$$
$$y(t) = y_0 e^t.$$

Example 2. For the damped harmonic oscillator

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0,$$

we found two (scalar) solutions

$$y_1(t) = e^{-t} \qquad \text{and} \qquad y_2(t) = e^{-2t}.$$

You should also recall that this second-order equation can be converted to a first-order system where

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -2y - 3v. \end{aligned}$$

Given a linear system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY},$$

how do we calculate the vector in the vector field at any given point $\mathbf{Y}_0$?

How do we calculate the equilibrium points of

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}?$$

Theorem. The origin is always an equilibrium point of a linear system. It is the only equilibrium point if and only if $\det \mathbf{A} \neq 0$.

The Linearity Principle

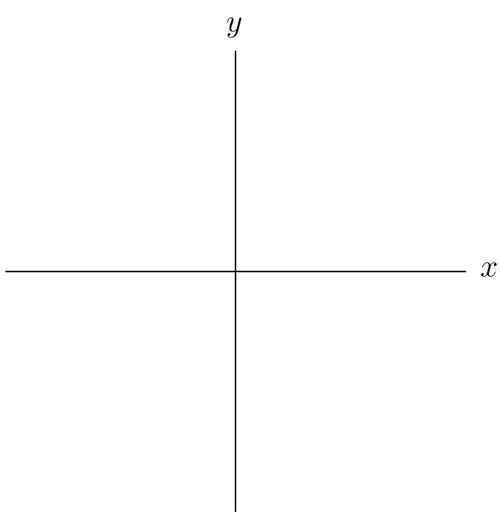
Let's return to Example 1. For practice, I'll use vector notation this time:

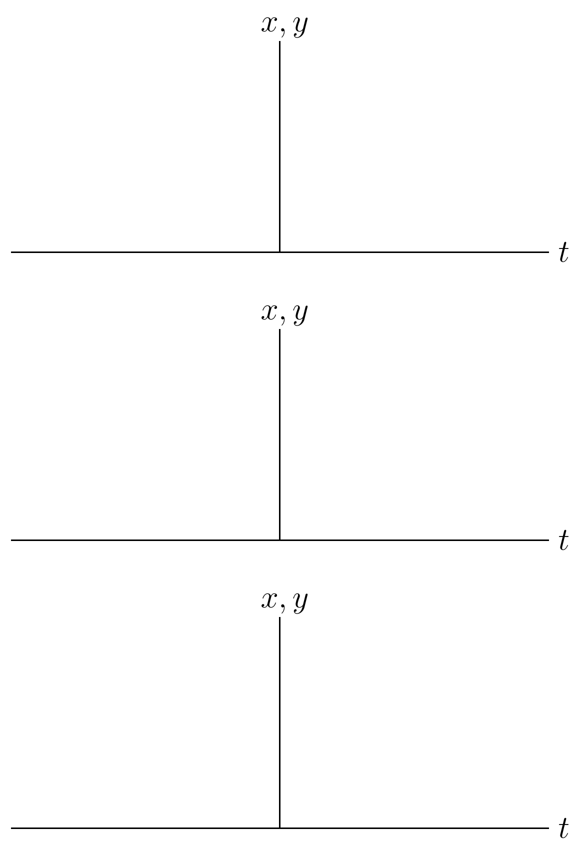
$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{Y}$$

Also consider three different initial conditions

$$\mathbf{Y}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \mathbf{Y}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \mathbf{Y}_3 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Let's see what happens when we graph the corresponding solutions.





How are these three solutions related?

Linearity Principle Suppose

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

is a linear system of differential equations.

1. If $\mathbf{Y}(t)$ is a solution of this system and k is any constant, then $k\mathbf{Y}(t)$ is also a solution.
2. If $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are two solutions of this system, then $\mathbf{Y}_1(t)+\mathbf{Y}_2(t)$ is also a solution.

Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{Y}$$

and the two solutions

$$\mathbf{Y}_1(t) = \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = \begin{pmatrix} e^{t} \\ e^{t} \end{pmatrix}.$$

Any linear combination of $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ is also a solution to the system.

We can use this observation to solve any initial-value problem involving this system.

Example. Solve

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

In general, how many solutions do we need to be able to solve any initial-value problem?