Method of integrating factors:

Summary from last class: Given a linear differential equation of the form

$$\frac{dy}{dt} + a(t)y = r(t),$$

the integrating factor (magic function) is

$$\mu(t) = e^{\left(\int a(t)\,dt\right)}.$$

Last class I went too fast at the end of the following example:

Example. $\dfrac{dy}{dt} = -2ty + 4e^{-t^2}$

We rewrite the equation as

$$\frac{dy}{dt} + 2ty = 4e^{-t^2},$$

and the integrating factor is

$$\mu(t) = e^{\int 2t\,dt} = e^{t^2}.$$

Multiplying both sides of the differential equation by $\mu(t)$ yields

$$\left(e^{t^2}\right)\left(\frac{dy}{dt}\right) + \left(2te^{t^2}\right)(y) = 4,$$

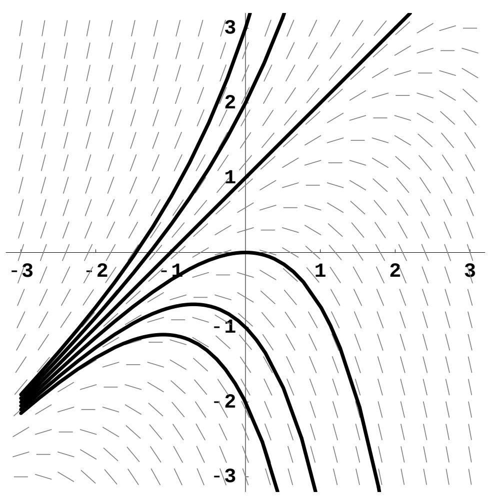
which is the same as the equation

$$\frac{d\left(e^{t^2}y\right)}{dt} = 4.$$

Here's how we find the general solution from this equation:

Let's do an example that I've mentioned before. It looks easier, but in some ways, it is actually more difficult.

Example. $\dfrac{dy}{dt} = y - t$



Linearity Principles

Why are linear equations so much more amenable to analytic techniques than nonlinear equations? The reason is that their solutions satisfy important linearity principles.

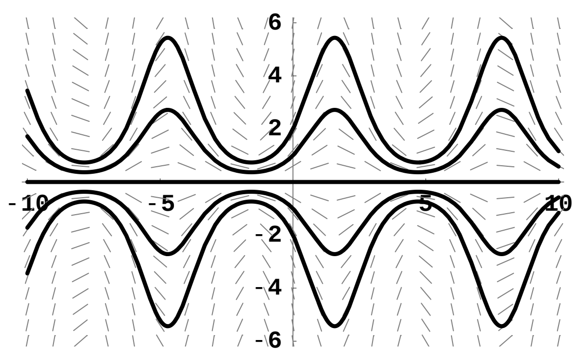
Let's begin with homogeneous linear equations:

Linearity Principle. If $y_1(t)$ is a solution of a homogeneous linear differential equation

$$\frac{dy}{dt} + a(t)y = 0,$$

then any *constant* multiple $y_k(t) = ky_1(t)$ of $y_1(t)$ is also a solution. In other words, given a constant $k \neq 1$ and a solution $y_1(t)$, we obtain another solution by multiplying $y_1(t)$ by k.

Example. $\frac{dy}{dt} = (\cos t)y$



Note that the Linearity Principle is not true for nonlinear equations. For example, consider

$$\frac{dy}{dt} = y^2.$$

Check that one solution is

$$y_1(t) = \frac{1}{1-t},$$

and then check that

$$y_2(t) = 2y_1(t) = \frac{2}{1-t}$$

is not a solution.

There is a similar "linearity" principle for nonhomogeneous linear equations:

Extended Linearity Principle For First-Order Equations. Consider a first-order, nonhomogeneous, linear equation

$$\frac{dy}{dt} + a(t)y = r(t)$$

and its associated homogeneous equation

$$\frac{dy}{dt} + a(t)y = 0.$$

1. If $y_h(t)$ is any solution of the homogeneous equation and $y_p(t)$ ("p" for particular solution) is *any* solution of the nonhomogeneous equation, then $y_h(t) + y_p(t)$ is also a solution of the nonhomogeneous equation.

2. Suppose $y_p(t)$ and $y_q(t)$ are two solutions of the nonhomogeneous equation. Then $y_p(t) - y_q(t)$ is a solution of the associated homogeneous equation.

Therefore, if $y_h(t)$ is nonzero, $ky_h(t) + y_p(t)$ is the general solution of the nonhomogeneous equation.

Example. Recall $\dfrac{dy}{dt} = y - t$.

We calculated the general solution

$$y(t) = 1 + t + ce^t.$$

Example. Recall $\dfrac{dy}{dt} = -2ty + 4e^{t^2}$.

We calculated the general solution

$$y(t) = 4te^{-t^2} + ce^{-t^2}.$$

The Extended Linearity Principle is the basis of another technique for solving nonhomogeneous linear equations—the Method of the Lucky Guess: