MA 226 March 15, 2004

Review for Linear Systems

Linearity Principle Suppose

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

is a linear system of differential equations.

1. If $\mathbf{Y}(t)$ is a solution of this system and k is any constant, then $k\mathbf{Y}(t)$ is also a solution.
2. If $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are two solutions of this system, then $\mathbf{Y}_1(t)+\mathbf{Y}_2(t)$ is also a solution.

"Straight-line" Solutions. Suppose that

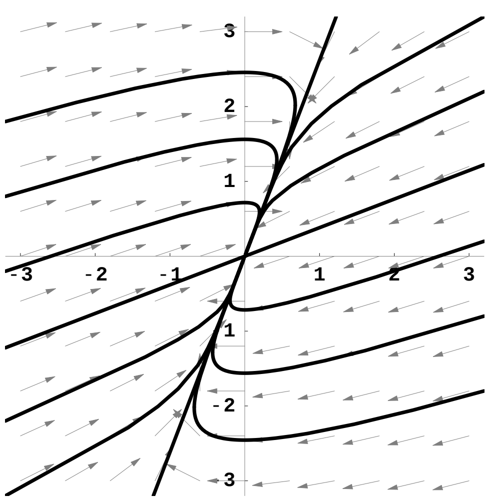
$$\mathbf{AY}_0 = \lambda\mathbf{Y}_0$$

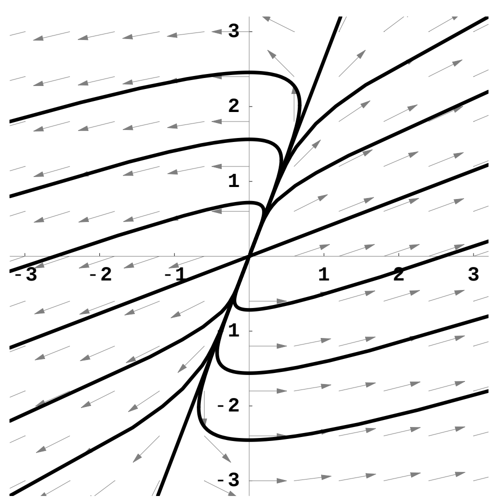
for some nonzero vector $\mathbf{Y}_0$ and some scalar λ. Then the function

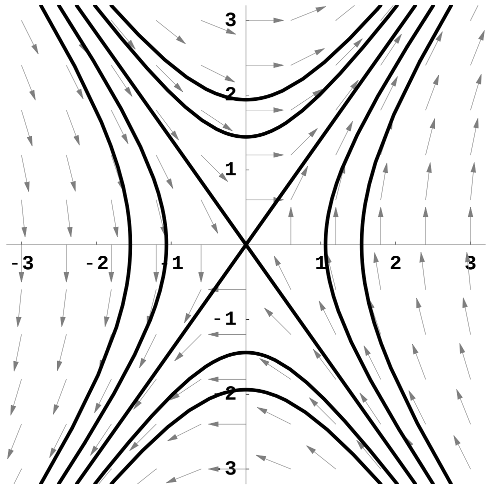
$$\mathbf{Y}(t) = e^{\lambda t}\mathbf{Y}_0$$

is a solution to the linear differential equation

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}.$$







What happens if the eigenvalues of the system are complex numbers?

Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \mathbf{Y}.$$

Let's see that happens if we take a look at this system using `MatrixFields` and then we'll compute the eigenstuff for this matrix.

We now have a complex-valued solution of the form

$$\mathbf{Y}_c(t) = e^{(-2+i)t} \begin{pmatrix} 2 \\ 1+i \end{pmatrix}.$$

We are interested in real-valued solutions. What good is this complex-valued solution?

Once again Euler comes to the rescue: Remember the power series for the exponential function? It is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots.$$

Let's use this series where $x = bi$.

We will use Euler's formula applied to the complex-valued function

$$e^{(a+bi)t}.$$

But why does this help us solve our differential equation?

Theorem. Consider

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y},$$

where $\mathbf{A}$ is a matrix with real entries. If $\mathbf{Y}_c(t)$ is a complex-valued solution, then both

$$\mathrm{Re}\mathbf{Y}_c(t) \quad \text{and} \quad \mathrm{Im}\mathbf{Y}_c(t)$$

are real-valued solutions, and they are linearly independent.