Last class we learned Euler's formula which tells us how to calculate the exponential of a complex number, and we derived the important complex-valued function

$$e^{(a+bi)t} = e^{at}(\cos bt + i \sin bt).$$

Now we apply this formula to the complex-valued straight-line solution we derived last class:

Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \mathbf{Y}.$$

The straight-line solution that we derived last class is

$$\mathbf{Y}_c(t) = e^{(-2+i)t} \begin{pmatrix} 2 \\ 1+i \end{pmatrix}.$$

But why does this help us solve our differential equation?

Theorem. Consider

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y},$$

where $\mathbf{A}$ is a matrix with real entries. If $\mathbf{Y}_c(t)$ is a complex-valued solution, then both

$$\mathrm{Re}\mathbf{Y}_c(t) \quad \text{and} \quad \mathrm{Im}\mathbf{Y}_c(t)$$

are real-valued solutions, and they are linearly independent.

Now we can derive the general solution to

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \mathbf{Y}$$

using the complex-valued solution

$$\mathbf{Y}_c(t) = e^{(-2+i)t} \begin{pmatrix} 2 \\ 1+i \end{pmatrix}.$$

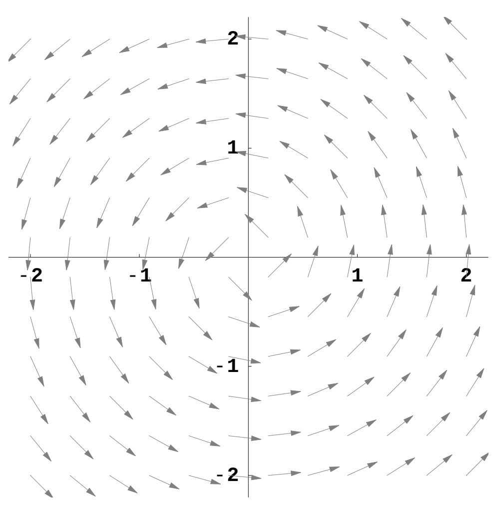
Three examples to illustrate the geometry of complex eigenvalues:

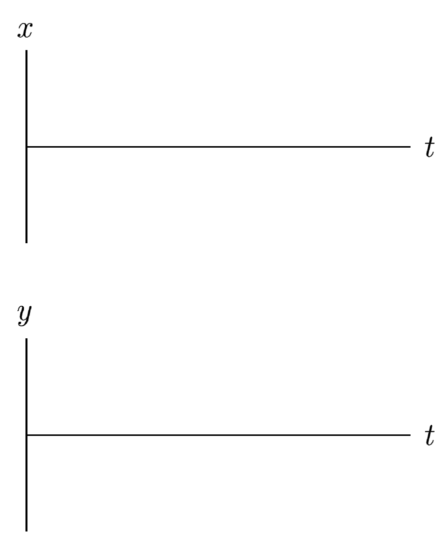
Example 1. $\dfrac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$ where

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The characteristic polynomial of $\mathbf{A}$ is $\lambda^2 + 1$, so the eigenvalues are $\lambda = \pm i$. One eigenvector associated to the eigenvalue $\lambda = i$ is

$$\mathbf{Y}_0 = \begin{pmatrix} i \\ 1 \end{pmatrix}.$$



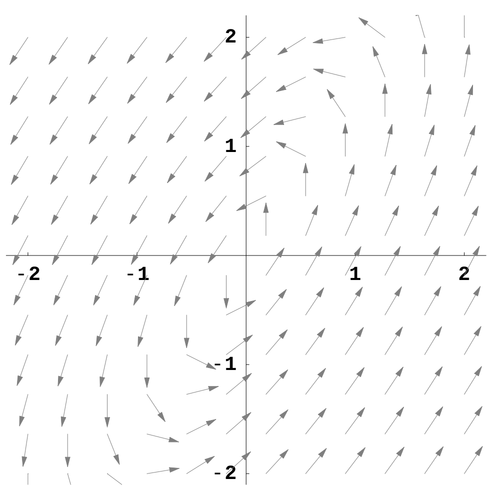


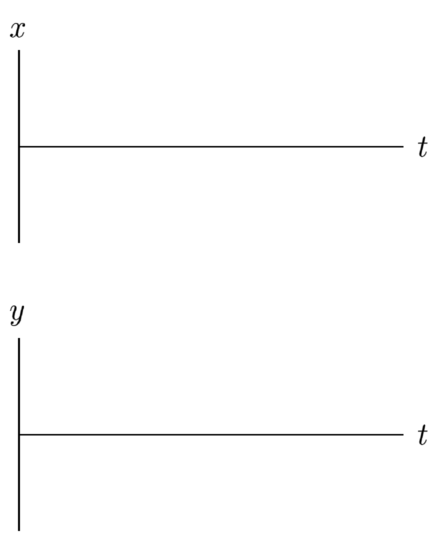
Example 2. $\dfrac{d\mathbf{Y}}{dt} = \mathbf{B}\mathbf{Y}$ where

$$\mathbf{B} = \begin{pmatrix} 2 & -2 \\ 4 & -2 \end{pmatrix}.$$

The characteristic polynomial of $\mathbf{B}$ is $\lambda^2 + 4$, so the eigenvalues are $\lambda = \pm 2i$. One eigenvector associated to the eigenvalue $\lambda = 2i$ is

$$\mathbf{Y}_0 = \begin{pmatrix} 1+i \\ 2 \end{pmatrix}.$$



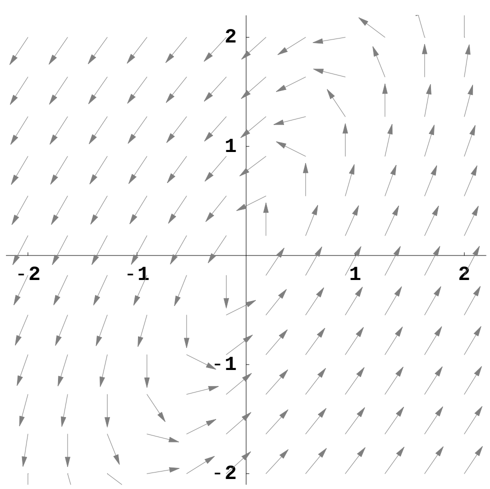


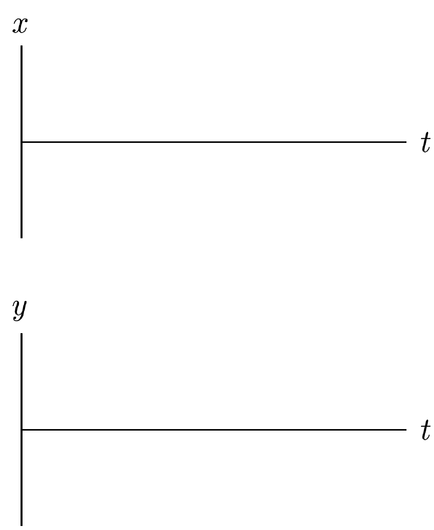
Example 3. $\dfrac{d\mathbf{Y}}{dt} = \mathbf{C}\mathbf{Y}$ where

$$\mathbf{C} = \begin{pmatrix} 1.9 & -2 \\ 4 & -2.1 \end{pmatrix}.$$

The characteristic polynomial of $\mathbf{C}$ is $\lambda^2 + 0.2\lambda + 4.01$, so the eigenvalues are $\lambda = -0.1 \pm 2i$. One eigenvector associated to the eigenvalue $\lambda = -0.1 + 2i$ is

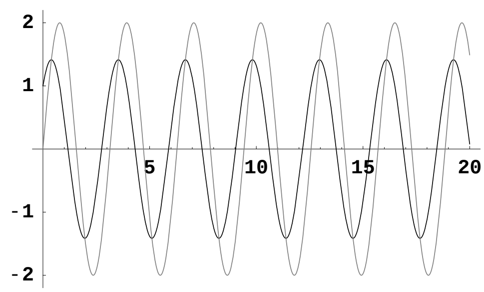
$$\mathbf{Y}_0 = \begin{pmatrix} 1+i \\ 2 \end{pmatrix}.$$



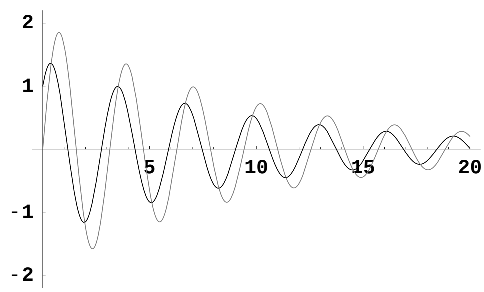


Summary: What information about solutions does a complex eigenvalue hold?

In Example 2, the eigenvalues are $\lambda = \pm 2i$. Here are the $x(t)$- and $y(t)$-graphs of a typical solution:



In Example 3, the eigenvalues are $\lambda = -0.1 \pm 2i$. Here are the $x(t)$- and $y(t)$-graphs of a typical solution:



In general, suppose that a linear system has complex eigenvalues $\lambda = a \pm bi$. What can you conclude about the solutions?