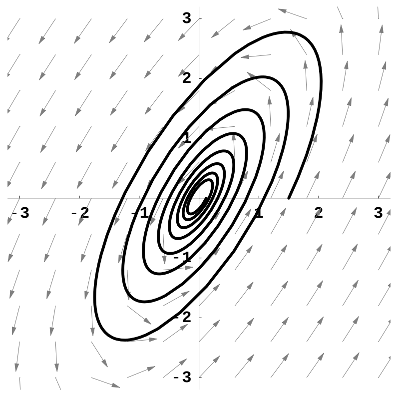
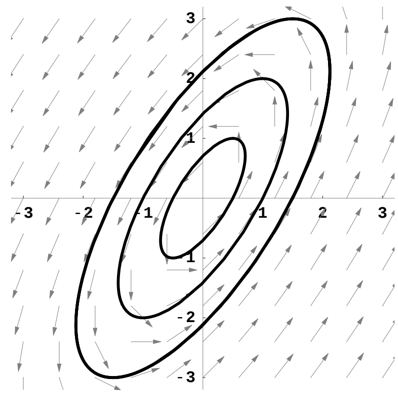
Summary: Linear systems with complex eigenvalues $\lambda = a \pm bi$

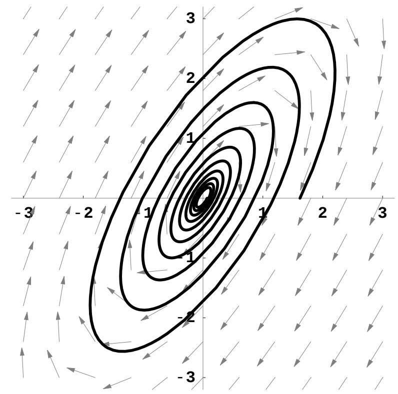
Here are the possible phase portraits:



spiral sink ($a < 0$)



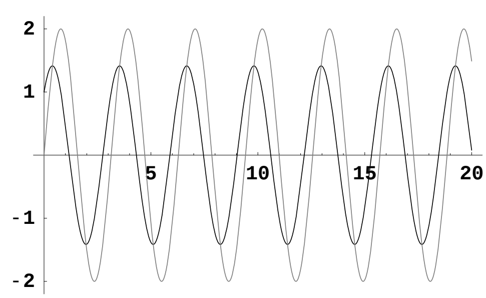
center ($a = 0$)



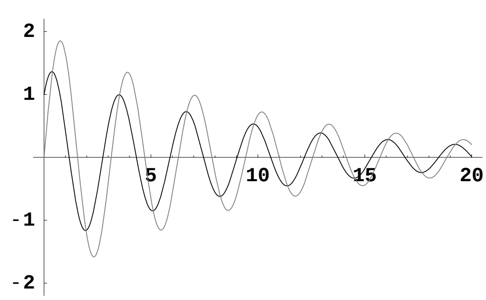
spiral source ($a > 0$)

What information can you get just from the complex eigenvalue alone?

Recall Example 2 from last class. The eigenvalues are $\lambda = \pm 2i$. Here are the $x(t)$- and $y(t)$-graphs of a typical solution:



In Example 3, the eigenvalues are $\lambda = -0.1 \pm 2i$. Here are the $x(t)$- and $y(t)$-graphs of a typical solution:



Frequency versus period: The solutions in Example 3 are not periodic in the strict sense. There is no time T such that

$$x(t+T) = x(t) \quad \text{and} \quad y(t+T) = y(t)$$

for all t. However, there is a period associated to these solutions. In the text, we call this the **natural period** of the solutions.

Perhaps it is best to think about these solutions as oscillating solutions that are decaying over time and to measure the oscillations in terms of their **frequency**.

Definition. The *frequency* F of an oscillating function $g(t)$ is the number of cycles that $g(t)$ makes in one unit of time.

Suppose that $g(t)$ is oscillating periodically with "period" T. What is its frequency F?

Example. Consider the standard sinusoidal functions $g(t) = \cos \beta t$ and $g(t) = \sin \beta t$.

Suppose we measure frequency in radians rather than in cycles. This measure of frequency is often called **angular frequency**. Let's denote the angular frequency by f. Then

$$f = 2\pi F.$$

Repeated eigenvalues: Sometimes the characteristic polynomial has the same real root twice. When this happens, we say that the eigenvalues are "repeated."

Example. $\dfrac{d\mathbf{Y}}{dt} = \mathbf{AY}$ where

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

The characteristic polynomial of $\mathbf{A}$ is $(\lambda - 2)^2$, so there is only one eigenvalue, $\lambda = 2$. Let's calculate the associated eigenvectors:

But we already know how to solve this system. How?

We obtain the general solution

$$\mathbf{Y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 e^{2t} + y_0 t e^{2t} \\ y_0 e^{2t} \end{pmatrix}$$

$$= e^{2t} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t e^{2t} \begin{pmatrix} y_0 \\ 0 \end{pmatrix}.$$

Note that this general solution is *not* written as a linear combination. Every nontrivial solution contains the first term, and most solutions contain both terms.

We use this result to motivate a different technique that we use to solve systems with repeated eigenvalues. We use a guessing technique where we guess a solution of the form

$$\mathbf{Y}(t) = e^{\lambda t}\mathbf{V}_0 + t e^{\lambda t}\mathbf{V}_1.$$

Note that the initial condition for this solution is $\mathbf{V}_0$.