

We apply what we have learned about linear systems to solve second-order homogeneous linear equations.

Let's return to the guessing technique for second-order equations that we learned about a month ago and see how it relates to what we have done with linear systems recently.

**Example.** Consider the equation

$$2\frac{d^2y}{dt^2} + 9\frac{dy}{dt} + 4y = 0.$$

1. Use a guessing technique to find two nonzero solutions  $y_1(t)$  and  $y_2(t)$  that are not multiples of each other.

2. Convert this equation to a first-order system and determine the analogous solutions  $\mathbf{Y}_1(t)$  and  $\mathbf{Y}_2(t)$ .

3. In what way are  $\mathbf{Y}_1(t)$  and  $\mathbf{Y}_2(t)$  special solutions?

Our goal today is to see how this guessing technique can be used to solve all second-order homogeneous equations.

Consider

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$$

with its characteristic equation

$$a\lambda^2 + b\lambda + c = 0$$

as well as the corresponding system

$$\begin{aligned}\frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -\frac{c}{a}y - \frac{b}{a}v\end{aligned}$$

with its characteristic equation

$$\det \begin{pmatrix} -\lambda & 1 \\ -\frac{c}{a} & -\frac{b}{a} - \lambda \end{pmatrix} = 0.$$

Useful observation: If  $\lambda$  is an eigenvalue, the vector

$$\mathbf{Y}_0 = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$$

is *always* an associated eigenvector.



3. One nonzero real eigenvalue  $\lambda$  of multiplicity two:

Conclusion: We can determine the general solution of a second-order homogeneous linear equation

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$$

immediately from the characteristic equation

$$a\lambda^2 + b\lambda + c = 0.$$

YOU DO NOT NEED TO CALCULATE THE EIGENVECTORS OR EVEN REDUCE TO A FIRST-ORDER SYSTEM if you simply want to produce the general solution of a second-order linear equation.

Application: We can apply what we have learned to the (damped) harmonic oscillator

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0.$$

In this case, we are assuming that the parameters  $m$  and  $k$  are positive and that  $b \geq 0$ . The characteristic equation

$$m\lambda^2 + b\lambda + k = 0$$

has eigenvalues

$$\frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$

There are three cases based on the value of the discriminant  $b^2 - 4mk$ .

1.  $b^2 - 4mk < 0$

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2.  $b^2 - 4mk = 0$

3.  $b^2 - 4mk > 0$

We can see the progression from underdamped to critically damped to overdamped with a Quicktime animation I have posted on the web site.