Recall the following example from last class.

Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{Y}$$

and the two solutions

$$\mathbf{Y}_1(t) = e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We can solve any initial-value problem for this differential equation using an appropriate linear combination of $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$. In other words, the general solution of this system is

$$k_1 e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The special solutions $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ come from the eigenvalues and eigenvectors of the matrix.

"Straight-line" Solutions. Suppose that

$$\mathbf{A}\mathbf{Y}_0 = \lambda \mathbf{Y}_0$$

for some nonzero vector $\mathbf{Y}_0$ and some scalar λ. Then the function

$$\mathbf{Y}(t) = e^{\lambda t} \mathbf{Y}_0$$

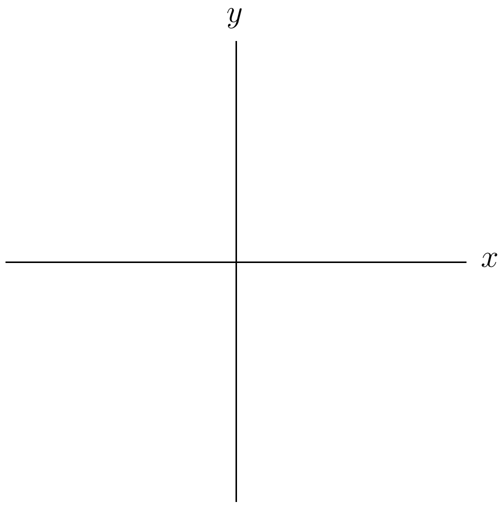
is a solution to the linear differential equation $d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y}$.

How do we find eigenvalues and eigenvectors given the matrix **A**?

Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 4 & -5 \\ -2 & 1 \end{pmatrix} \mathbf{Y}.$$

First let's see what `MatrixFields` tells us about the eigenvalues and eigenvectors of the matrix **A**.



Aside from the theory of algebraic linear equations

For what matrices **B** does the equation $\mathbf{BY} = \mathbf{0}$ have nontrivial solutions?

Singular Matrices. The matrix equation

$$\mathbf{BY} = \mathbf{0}$$

has nontrivial solutions **Y** if and only if $\det \mathbf{B} = 0$.

Note: Most matrices are nonsingular.

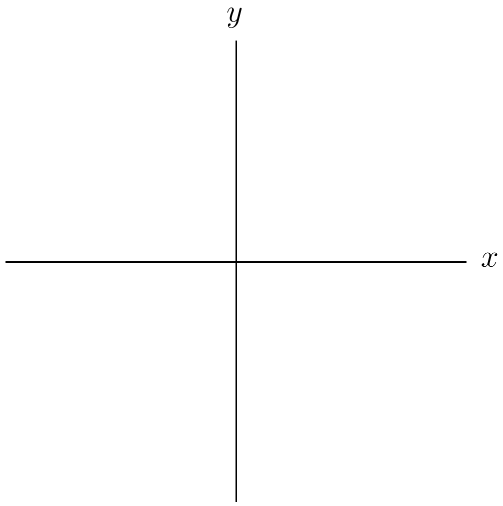
Now let's use this theorem to find eigenvalues and eigenvectors:

Back to the example:

Example. Find the general solution to

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 4 & -5 \\ -2 & 1 \end{pmatrix} \mathbf{Y}.$$

Using `HPGSystemSolver`, we plot the phase portrait for this system.



Facts about eigenvalues and eigenvectors: Given a 2×2 matrix $\mathbf{A}$,

1. The characteristic equation can have two real roots, one real root of multiplicity two, or two complex conjugate roots.
2. Given an eigenvector $\mathbf{Y}_0$ associated to an eigenvalue λ, then any nonzero scalar multiple $\mathbf{Y}_0$ is also an eigenvector associated to λ.
3. Eigenvectors associated to distinct eigenvalues are linearly independent.

Summary of Case of Two Distinct Real Eigenvalues

Suppose **A** is a matrix with two eigenvalues λ_1 and λ_2. To be consistent, we will assume that $\lambda_1 < \lambda_2$, that $\mathbf{V}_1$ is an eigenvector associated to λ_1, and that $\mathbf{V}_2$ is an eigenvector associated to λ_2. The general solution of

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$$

is

$$\mathbf{Y}(t) = k_1 e^{\lambda_1 t}\mathbf{V}_1 + k_2 e^{\lambda_2 t}\mathbf{V}_2.$$

Case 1: $\lambda_1 < \lambda_2 < 0$.