MA 226 April 12, 2004

More on linearization:

Example. Recall the (undamped) pendulum

$$\frac{d^2\theta}{dt^2} + \sin\theta = 0.$$

The corresponding system is

$$\begin{aligned} \frac{d\theta}{dt} &= v \\ \frac{dv}{dt} &= -\sin\theta. \end{aligned}$$

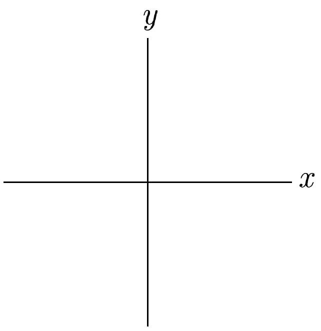
Last class we discussed linearization near the equilibrium point $(\pi, 0)$. What about near $(0, 0)$?

The following example is more typical of what happens when the eigenvalues are purely imaginary.

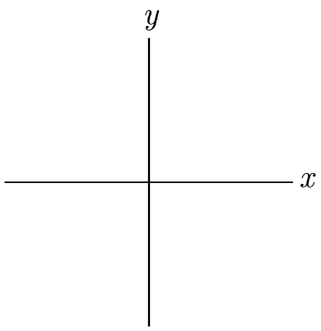
Example. Consider the one-parameter family of systems

$$\begin{aligned} \frac{dx}{dt} &= -y + \alpha x(x^2 + y^2) \\ \frac{dy}{dt} &= x + \alpha y(x^2 + y^2) \end{aligned}$$

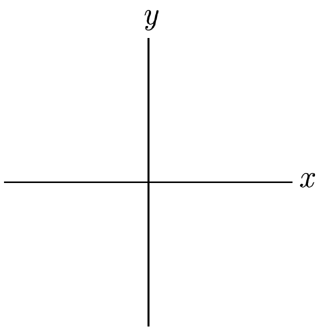
where α is a parameter. Note that $(0, 0)$ is always an equilibrium point.



$\alpha < 0$



$\alpha = 0$



$\alpha > 0$

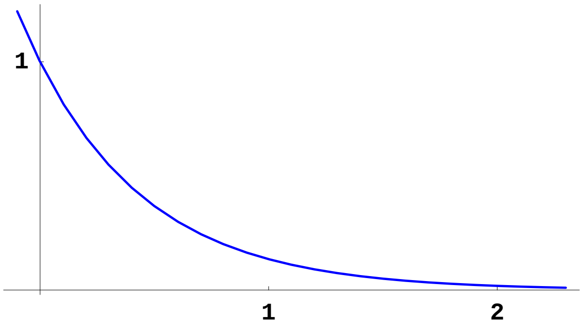
The Laplace transform

For the remainder of the semester, we are going to take a somewhat different approach to the solution of differential equations. We are going to study a way of transforming differential equations into algebraic equations.

We begin with a little review of improper integrals.

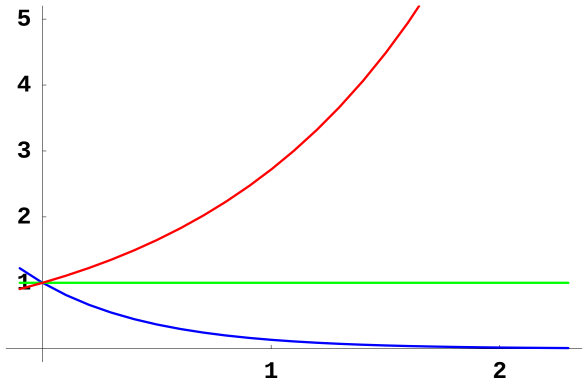
Example. Consider the improper integral

$$\int_0^\infty e^{-2t}\,dt.$$



Example. Consider the improper integrals

$$\int_0^\infty e^{-st}\,dt.$$



Definition. The *Laplace transform* of the function $y(t)$ is the function

$$Y(s) = \int_0^\infty y(t)\, e^{-st}\, dt.$$

This transform is an "operator" (a function on functions). It transforms the function $y(t)$ into the function $Y(s)$.

Notation: We often represent this operator using the script letter $\mathcal{L}$. In other words,

$$\mathcal{L}[y] = Y.$$

For example,

$$\mathcal{L}[1] = \frac{1}{s}.$$

Note that even if $y(t)$ is defined for all t, the Laplace transform $Y(s)$ may not be defined for all s.

Examples. Using *Mathematica* to calculate the improper integrals, we see that:

$$\mathcal{L}[\sin t] = \frac{1}{s^2+1} \quad \text{for} \quad s > 0,$$