Sinusoidal forcing

Today we are going to study forced equations where the forcing function is sinusoidal (either sine or cosine). But first I want to do a calculation that saves a little time in these "guessing" problems.

A time saver: There's a calculation that we've already done twice before. It is also useful for guessing $y_p(t)$. Consider the function $y(t) = e^{\lambda t}$ and calculate

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky.$$

Conclusion: If we guess

$$y(t) = ae^{\lambda t}$$

where a is a constant, we get

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = a\,p(\lambda)\,e^{\lambda t}$$

where $p(\lambda) = m\lambda^2 + b\lambda + k$ is the characteristic polynomial.

Example. Let's calculate the general solution to the equation

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + 2y = \cos 2t.$$

We can see the implications of this computation by entering this equation into `MassSpring` on the CD.

A little translation:

Consider the second-order linear forced equation

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = f(t)$$

where m, b, and k are all positive.

Engineering terminology:

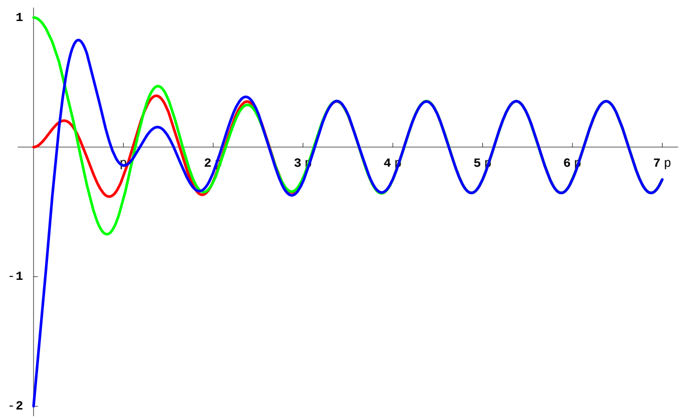
forced response—any solution to the forced equation.

steady-state response—behavior of the forced response over the long term.

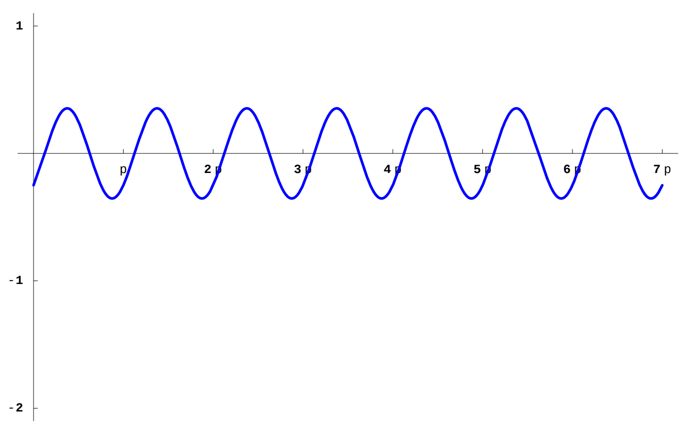
natural (or free) response—any solution of the associated homogeneous equation.

Why are initial conditions essentially irrelevant?

Here are the graphs of three solutions:



Here is the graph of the steady-state solution:



When we guess a solution of the form $y_p(t) = ae^{i\omega t}$ and compute the complex number a, we have essentially determined everything we need to know about the steady-state solution. Euler's formula gives us a nice way of determining the amplitude, frequency, and phase angle of the steady-state solution immediately from a: