More on Laplace transforms and second-order equations

The second-order equations that we discussed last class were undamped. In order to consider the full range of second-order equations, we need one more property of the transform.

Shifting the s-axis. Let $Y(s)$ denote $\mathcal{L}[y(t)]$. Then

$$\mathcal{L}[e^{at}\, y(t)] =$$

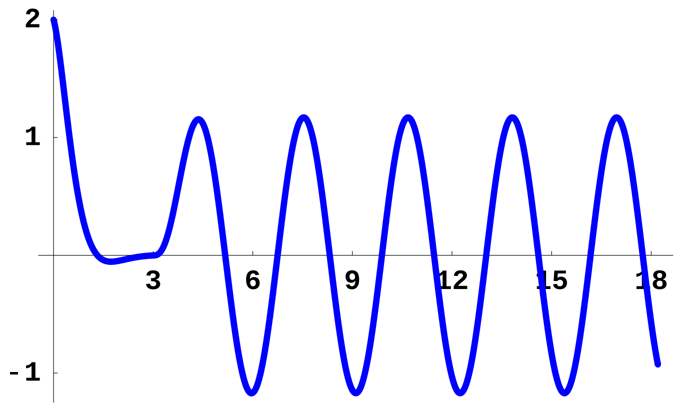
Example 1. Calculate $\mathcal{L}[e^{-2t} \cos 3t]$.

Example 2. Calculate $\mathcal{L}^{-1}\left[\frac{2s+7}{s^2+4s+7}\right]$.

Let's solve the initial-value problem

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 7y = 10\,u_3(t)\ \sin 2(t-3), \quad y(0) = 2,\ y'(0) = -1.$$

Before we get too far into the messy formulas, let's look at the graph of the solution using `HPGSystemSolver`:



Now for the formulas:

1. Transform both sides of the equation:

2. Solve for $\mathcal{L}[y]$:

3. Calculate the inverse Laplace transform:

We calculated

$$\mathcal{L}^{-1}\left[\frac{2s+7}{s^2+4s+7}\right] = 2e^{-2t}\cos\sqrt{3}\,t + \sqrt{3}e^{-2t}\sin\sqrt{3}\,t$$

in Example 2.

To invert the second term, we take advantage of some algebra done before class:

(a) Partial fractions decomposition:

$$\frac{1}{(s^2+4)(s^2+4s+7)} = \frac{1}{73}\left(\frac{4s+13}{s^2+4s+7} - \frac{4s-3}{s^2+4}\right)$$

(b) Inverse related to the first term:

$$\mathcal{L}^{-1}\left[\frac{4s+13}{s^2+4s+7}\right] = 4e^{-2t}\cos\sqrt{3}\,t + \frac{5\sqrt{3}}{3}e^{-2t}\sin\sqrt{3}\,t$$

(c) Inverse related to the second term:

$$\mathcal{L}^{-1}\left[\frac{4s-3}{s^2+4}\right] = 4\cos 2t - \frac{3}{2}e^{-2t}\sin 2t$$

After we put all of this together, we get the solution

$$y(t) = 2e^{-2t}\cos\sqrt{3}\,t + \sqrt{3}e^{-2t}\sin\sqrt{3}\,t +$$

$$\frac{20}{73}u_3(t)\left(4e^{-2(t-3)}\cos\sqrt{3}(t-3) + \frac{5\sqrt{3}}{3}e^{-2(t-3)}\sin\sqrt{3}(t-3) - 4\cos 2(t-3) + \frac{3}{2}\sin 2(t-3)\right)$$

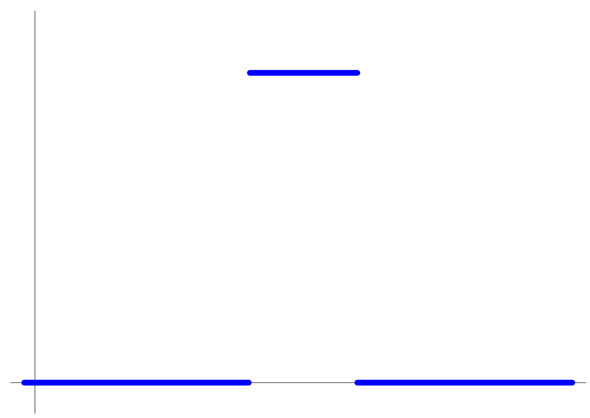
Impulse forcing

It is convenient to use Laplace transforms to solve linear differential equations with discontinuous forcing, but there are other, less convenient, ways to solve those equations. Now we consider equations that cannot be solved without the use of transforms (as far as I know).

Dirac Delta Function. The Dirac delta "function" $\delta_a(t)$ is used to model impulse forcing.

Suppose we want to model a unit force that is applied instantaneously at time $t = a$. We begin with the function

$$g_{\Delta t}(t) = \begin{cases} \dfrac{1}{\Delta t}, & \text{if } a - \dfrac{\Delta t}{2} \leq t < a + \dfrac{\Delta t}{2}; \\ 0, & \text{otherwise.} \end{cases}$$



We can write $g_{\Delta t}$ in terms of the Heaviside function. We get

$g_{\Delta t} =$

Now let's calculate the Laplace transform of $g_{\Delta t}$:

Finally we take the limit as $\Delta t \to 0$.

Dirac Delta Function. The Dirac delta function $\delta_a(t)$ is the "function" such that

$$\mathcal{L}[\delta_a] = e^{-as}.$$