On Friday we studied sinusoidal forcing with damping. Today we discuss sinusoidal forcing in the absense of damping.

First, there is one piece of unfinished business from last class. We calculated the steady-state solution

$$y_p(t) = -\tfrac{1}{4}(\cos 2t - \sin 2t)$$

for the equation

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + 2y = \cos 2t,$$

and we did so using computations that involved complex numbers. In fact, we found $y_p(t)$ as the real part of

$$y_c(t) = a\, e^{(2i)t}$$

where

$$a = -\tfrac{1}{4}(1 + i).$$

Now I want to explain why the polar representation of a (see pp. 713–715 in Appendix B) tells us everything we want to know about the steady-state solution.

Sinusoidal forcing without damping

Consider sinusoidal forcing in the absense of damping, for example, the mass-spring system without the dashpot.

Example. Let's find the general solution to

$$\frac{d^2y}{dt^2} + 3y = \cos \omega t.$$

Note the lack of a damping term. We want to see what happens with various forcing frequencies.

Unfortunately the parts of the solution that correspond to the associated homogeneous equation do not die out. So to get some qualitative understanding in this case, we make a simplifying assumption. We consider the solution that satisfies the initial condition $(y(0), y'(0)) = (0, 0)$.

Trig identity:

$$\cos at - \cos bt = -2\,(\sin \alpha t)\,(\sin \beta t)$$

where

$$\alpha = \frac{a+b}{2} \quad \text{and} \quad \beta = \frac{a-b}{2}.$$

The number α is the average of a and b, and β is called the *half-difference* of a and b.

Example. Let's use this trig identity to get a rough idea of the graph of

$$\cos \omega t - \cos \sqrt{3}\, t$$

where $\omega = 1.6$.

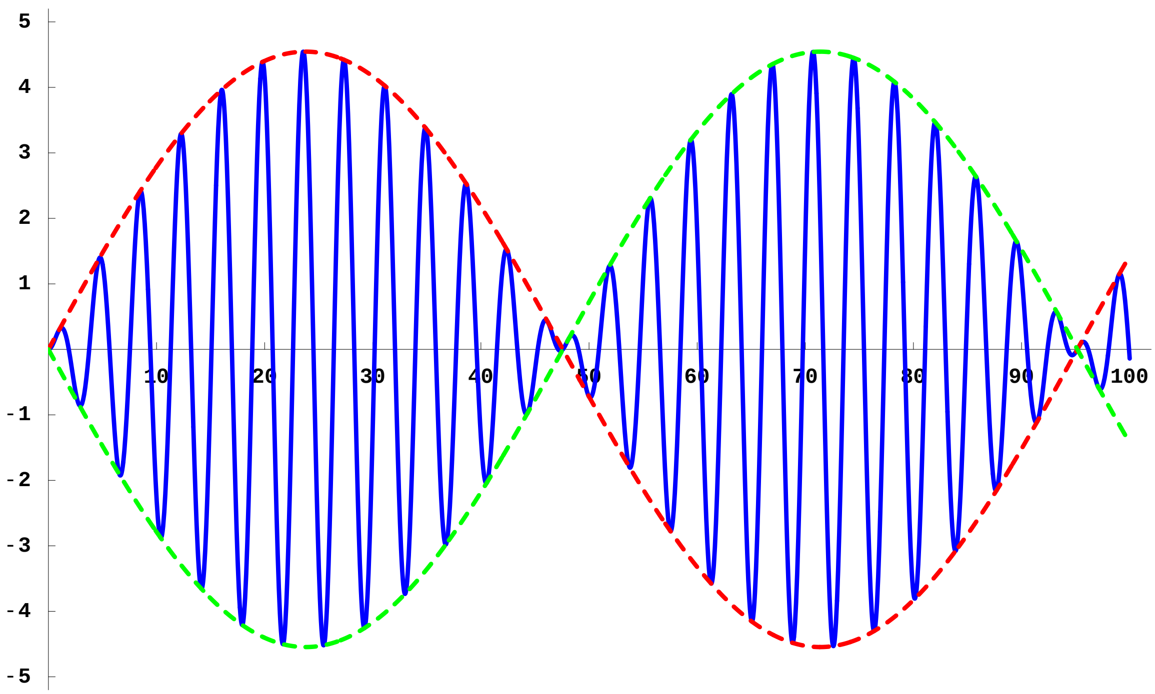
Using this trig identity, we can rewrite our solution as

$$y(t) = \frac{-2}{3-\omega^2}\,(\sin \alpha t)\,(\sin \beta t)$$

where

$$\alpha = \frac{\omega + \sqrt{3}}{2} \quad \text{and} \quad \beta = \frac{\omega - \sqrt{3}}{2}.$$

Here is the graph of this solution in the case where $\omega = 1.6$.



What happens if $\omega = \sqrt{3}$?

Example.

$$\frac{d^2y}{dt^2} + 3y = \cos\sqrt{3}\,t$$

The complexified equation is

$$\frac{d^2y}{dt^2} + 3y = e^{i\sqrt{3}\,t}.$$

What guess should we use?