The vector field of an autonomous system

We get a better geometric understanding of the solutions of a first-order system if we convert the system into a vector equation.

Example 1. Once again we consider a simple mass-spring system

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -y.$$

By some intelligent guessing we know a few solutions. One is

$$(y(t), v(t)) = (\cos t, -\sin t).$$

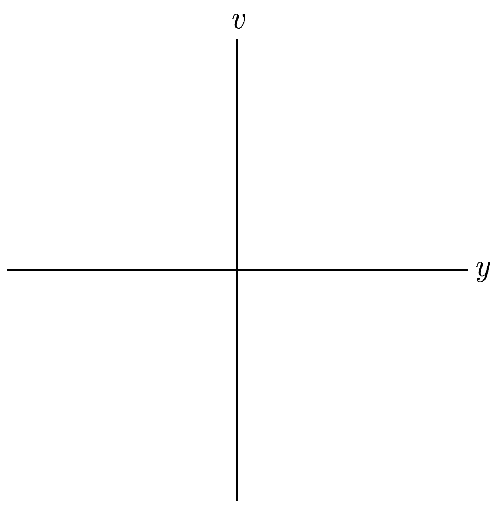
We rewrite this system and the solution in terms of vectors:

Now for the geometric interpretation of

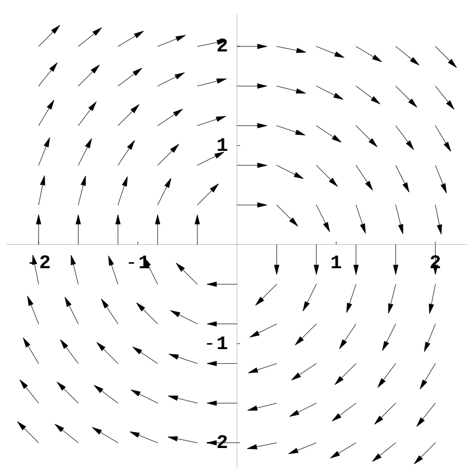
$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y}),$$

where

$$\mathbf{F}(\mathbf{Y}) = \mathbf{F}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} v \\ -y \end{pmatrix}.$$



The direction field associated with this system is



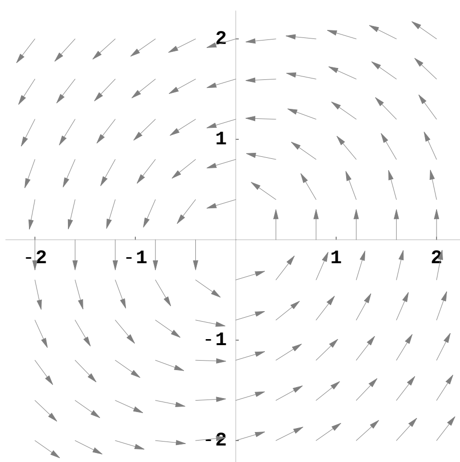
Example 2. Consider the system

$$\frac{dx}{dt} = -y$$
$$\frac{dy}{dt} = x - 0.3y.$$

The vector field associated with this system is

$$\mathbf{F}(\mathbf{Y}) = \mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x - 0.3y \end{pmatrix}.$$

Here's the direction field:

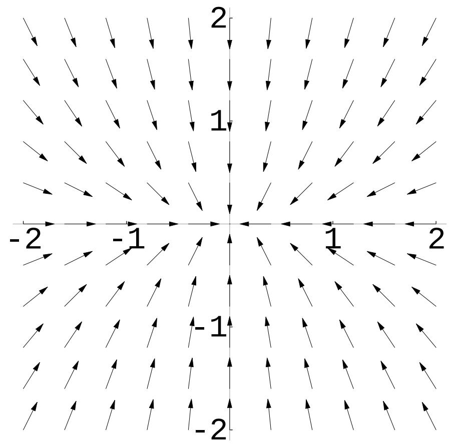


Consider the following 8 first-order systems:

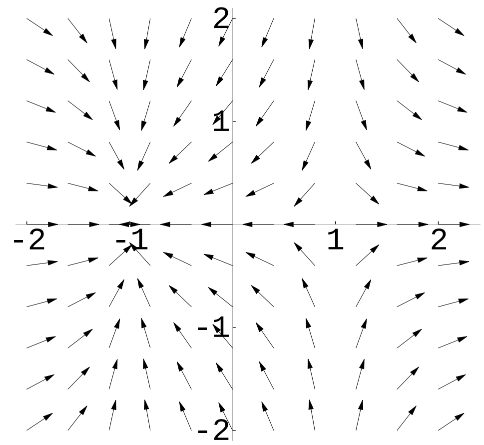
1. $\dfrac{dx}{dt} = -x$, $\dfrac{dy}{dt} = y^2 - 1$
2. $\dfrac{dx}{dt} = -2x$, $\dfrac{dy}{dt} = -y$
3. $\dfrac{dx}{dt} = -x - 2y$, $\dfrac{dy}{dt} = y$
4. $\dfrac{dx}{dt} = 1 - y$, $\dfrac{dy}{dt} = 1 + x$
5. $\dfrac{dx}{dt} = x$, $\dfrac{dy}{dt} = 2x - y$
6. $\dfrac{dx}{dt} = y - 1$, $\dfrac{dy}{dt} = -1 - x$
7. $\dfrac{dx}{dt} = -x$, $\dfrac{dy}{dt} = -2y$
8. $\dfrac{dx}{dt} = x^2 - 1$, $\dfrac{dy}{dt} = -y$

Four of the associated direction fields are shown below. Pair the direction fields with their associated systems. Provide a brief justification for your choice.

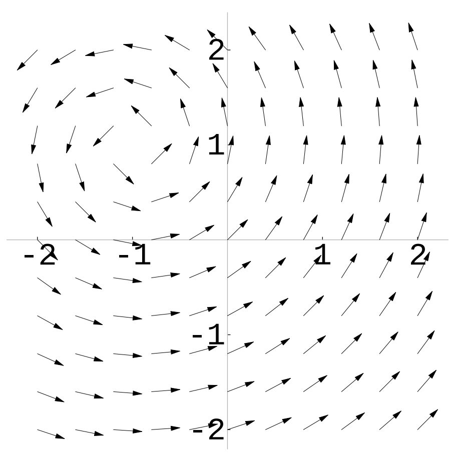
Direction Field A



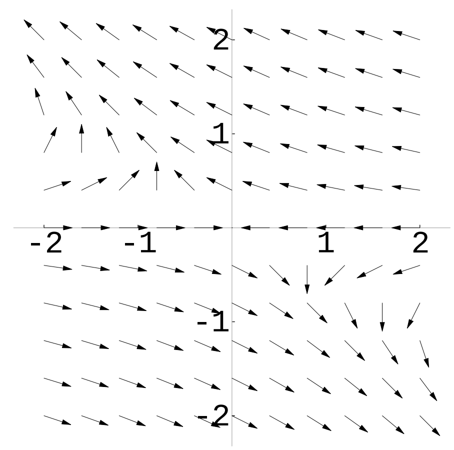
Direction Field B



Direction Field C



Direction Field D



Analytic Techniques:

There are few analytic techniques that work for both linear and nonlinear systems.

1. You can always check to see if a given function is a solution (no wrong answers).

   For example, consider the initial-value problem

$$\frac{dx}{dt} = 2y - x \qquad \qquad (x_0, y_0) = (2, 1).$$
$$\frac{dy}{dt} = y$$

   Using the vector notation

$$\mathbf{Y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

   we can write this IVP as

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2y - x \\ y \end{pmatrix}, \qquad \mathbf{Y}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

   Claim: The function

$$\mathbf{Y}(t) = \begin{pmatrix} e^t + e^{-t} \\ e^t \end{pmatrix}$$

   solves the IVP.