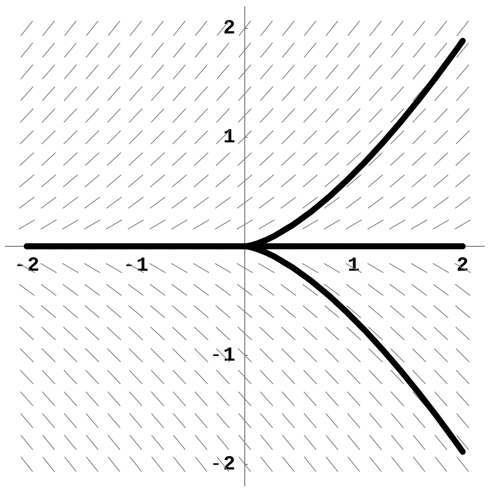
Quick Recap of Example of Nonuniqueness

At the end of last class, we briefly discussed an example that lacked uniqueness.

Example. $\frac{dy}{dt} = \sqrt[3]{y}$



After separating variables, we were able to derive three solutions that satisfied the initial condition $y(0) = 0$.

$$y_1(t) = 0 \quad \text{for all } t$$

$$y_2(t) = \begin{cases} 0 & \text{for } t < 0; \text{ and} \\ (\frac{2}{3}t)^{3/2} & \text{for } t \geq 0. \end{cases}$$

$$y_3(t) = \begin{cases} 0 & \text{for } t < 0; \text{ and} \\ -(\frac{2}{3}t)^{3/2} & \text{for } t \geq 0. \end{cases}$$

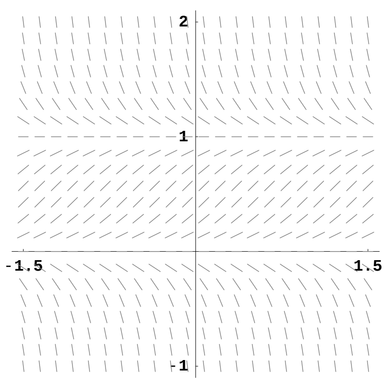
Why doesn't this example violate the Uniqueness Theorem?

Applications of Uniqueness

The Uniqueness Theorem has many useful consequences. Here are three examples:

Example 1. $\dfrac{dy}{dt} = -2ty^2$

Example 2. $\dfrac{dy}{dt} = 4y(1-y)$



Example 3. $\dfrac{dy}{dt} = e^t \sin y$

Autonomous Differential Equations

A first-order differential equation with independent variable t and dependent variable y is **autonomous** if

$$\frac{dy}{dt} = f(y).$$

The rate of change of $y(t)$ depends only on the value of y.

Examples of autonomous equations: exponential growth model, radioactive decay, logistic population model

Example. $\frac{dv}{dt} = -kv + a \sin bt$

This is a nonautonomous linear differential equation that is related to simple models of voltage in an electric circuit (k, a, and b are parameters).

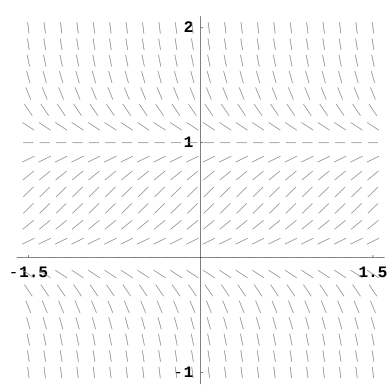
Comments:

1. Many interesting models in science and engineering are autonomous (but not every model).

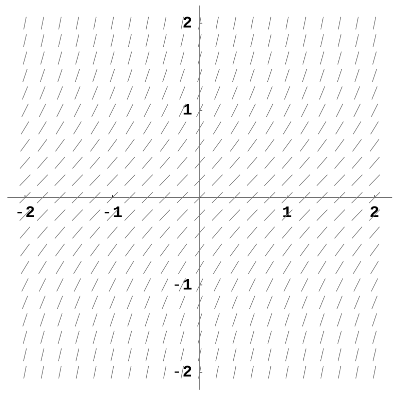
2. Every autonomous equation is separable, but the integrals may be impossible to calculate in terms of standard functions.

Basic Fact: Given the graph of one solution to an autonomous equation, we can get the graphs of many other solutions by translating that graph left or right.

Example 1. $\frac{dy}{dt} = 4y(1-y)$

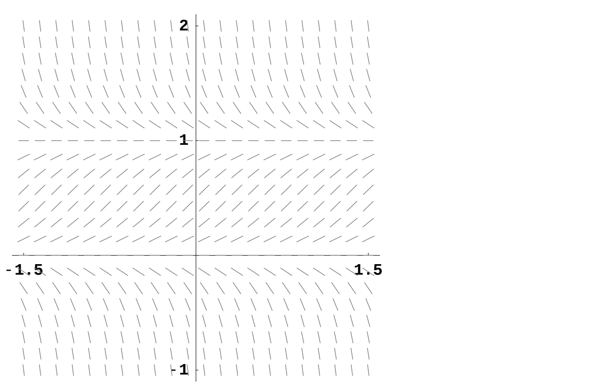


Example 2. $\dfrac{dy}{dt} = 1 + y^2$



The slope field has so much redundant information that we can replace it with the **phase line**. Here's the phase line for our standard example:

Example. $\dfrac{dy}{dt} = 4y(1 - y)$



How do we go about building a phase line from a differential equation?

Example. $\dfrac{dy}{dt} = y^2 \cos y$