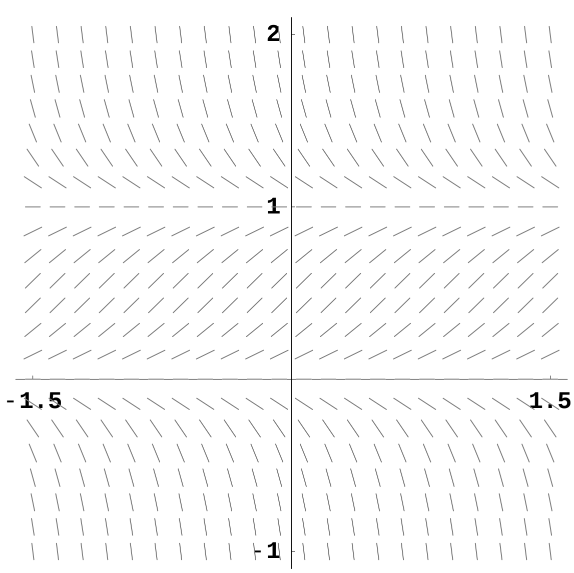
Phase lines

Last class we talked about how one solution for an autonomous equation

$$\frac{dy}{dt} = f(y)$$

yields infinitely many solutions by translating its graph left or right. As a result, one can get a good qualitative understanding of the solutions using only the **phase line**.

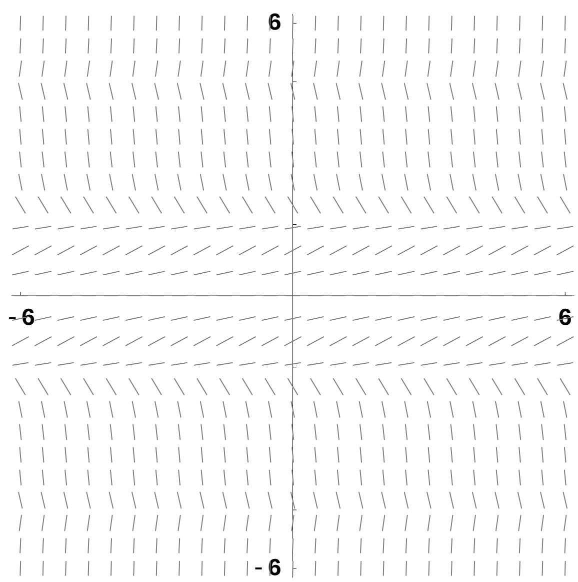
Example. $\dfrac{dy}{dt} = 4y(1-y)$



Professor Devaney built a simple Quicktime animation that illustrates how you should interpret this phase line. There is a link to it on our course web page.

How do we go about building a phase line from a differential equation?

Example. $\dfrac{dy}{dt} = y^2 \cos y$



You should also learn how to produce a phase line from the graph of $f(y)$ as well as rough graphs of solutions from a phase line.

Linear Differential Equations

A first-order differential equation

$$\frac{dy}{dt} = f(t, y)$$

with independent variable t and dependent variable y is **linear** if it can be written as

$$\frac{dy}{dt} = g(t)y + r(t).$$

In other words, the *dependent* variable only appears linearly in the equation.

Linear differential equations:

$$\frac{dy}{dt} = 5y$$

$$\frac{dy}{dt} = (\cos t)y$$

$$\frac{dy}{dt} = y - t^2$$

Nonlinear differential equations:

$$\frac{dy}{dt} = t \cos y$$

$$\frac{dy}{dt} = y^2 - t$$

The linear differential equation

$$\frac{dy}{dt} = g(t)y + r(t)$$

is **homogeneous** if $r(t) = 0$ for all t. Otherwise, it is **nonhomogeneous**. (Some people use the term inhomogeneous.)

Where have we seen homogeneous linear differential equations before?

Method of integrating factors:

For this method we rewrite the nonhomogeneous equation as

$$\frac{dy}{dt} + a(t)y = r(t).$$

In other words, $a(t) = -g(t)$. There is no mathematical significance to this step. It just avoids an annoying minus sign in one of the formulas.

Now for some magic:

Summary: Given a linear differential equation of the form

$$\frac{dy}{dt} + a(t)y = r(t),$$

the integrating factor (magic function) is

$$\mu(t) = e^{\left(\int a(t)\,dt\right)}.$$

Example. $\dfrac{dy}{dt} = -2ty + 4e^{-t^2}$