Linear systems—a brief review

A linear system (with constant coefficients) can be written as

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y},$$

where $\mathbf{A}$ is a square matrix of constants (the coefficients). For us, $\mathbf{A}$ will be a 2×2 matrix. Using the Linearity Principle, we can produce many solutions from just a few:

If $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are solutions, then

$$k_1\mathbf{Y}_1(t) + k_2\mathbf{Y}_2(t)$$

is a solution for any choice of constants k_1 and k_2.

Now recall the example we did just before break:

Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}\mathbf{Y}$$

and the two solutions

$$\mathbf{Y}_1(t) = \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = \begin{pmatrix} e^{t} \\ e^{t} \end{pmatrix}.$$

Any linear combination of $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ is also a solution to the system.

To solve the IVP

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}\mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} -1 \\ -2 \end{pmatrix},$$

we find k_1 and k_2 such that

$$k_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

We determined that $k_2 = -2$ and $k_1 = 1$. Therefore, the desired solution is

$$\mathbf{Y}(t) = 1\,\mathbf{Y}_1(t) - 2\,\mathbf{Y}_2(t),$$

which produces the solution

$$\begin{aligned} \mathbf{Y}(t) &= e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 2e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} e^{-t} - 2e^{t} \\ -2e^{t} \end{pmatrix}. \end{aligned}$$

Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{Y}$$

and the two solutions

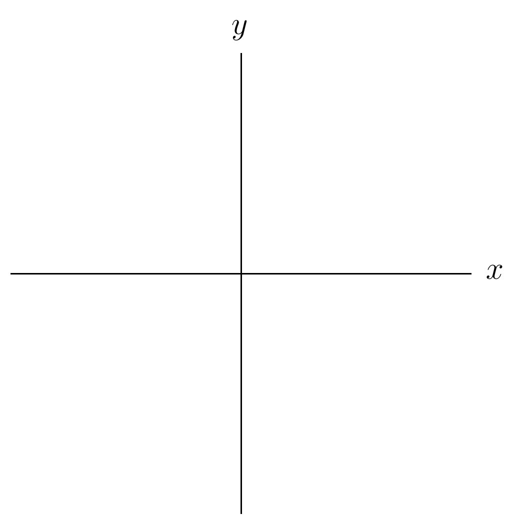
$$\mathbf{Y}_1(t) = e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We can solve any initial-value problem for this differential equation using an appropriate linear combination of $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$. In other words, the general solution of this system is

$$k_1 e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The special solutions $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ come from the eigenvalues and eigenvectors of the matrix.

What geometric property characterizes $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$?



"Straight-line" Solutions. Suppose that

$$\mathbf{A}\mathbf{Y}_0 = \lambda \mathbf{Y}_0$$

for some nonzero vector $\mathbf{Y}_0$ and some scalar λ. Then the function

$$\mathbf{Y}(t) = e^{\lambda t} \mathbf{Y}_0$$

is a solution to the linear differential equation $d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y}$.

How do we find eigenvalues and eigenvectors given the matrix $\mathbf{A}$?

Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 4 & -5 \\ -2 & 1 \end{pmatrix} \mathbf{Y}.$$

First let's see what `MatrixFields` tells us about the eigenvalues and eigenvectors of the matrix $\mathbf{A}$.