A brief review from last class

To solve a linear system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$$

where $\mathbf{A}$ is a 2×2 matrix, we use the Linearity Principle applied to two *linearly independent* solutions $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$. The general solution is

$$k_1\mathbf{Y}_1(t) + k_2\mathbf{Y}_2(t).$$

Question: How do we find two linearly independent solutions?

Answer: We look for solutions whose solution curves consist of half-lines from the origin in the phase plane.

"Straight-line" Solutions. Suppose that

$$\mathbf{A}\mathbf{Y}_0 = \lambda\mathbf{Y}_0$$

for some nonzero vector $\mathbf{Y}_0$ and some scalar λ. Then the function

$$\mathbf{Y}(t) = e^{\lambda t}\mathbf{Y}_0$$

is a solution to the linear differential equation $d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y}$.

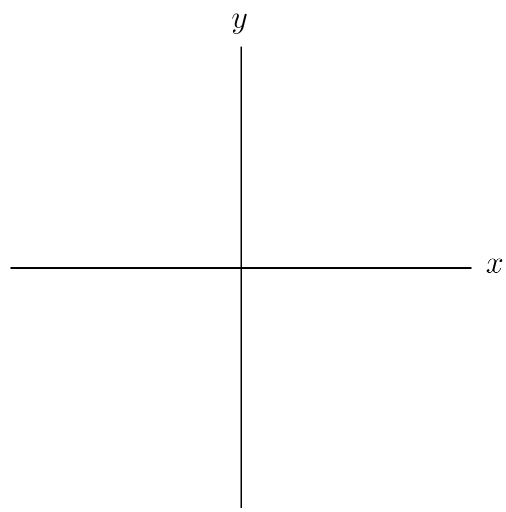
Terminology: The scalar λ is called an *eigenvalue* of the matrix $\mathbf{A}$ and the vector $\mathbf{Y}_0$ is called an *eigenvector* associated to the eigenvalue λ.

How do we find eigenvalues and eigenvectors given the matrix $\mathbf{A}$?

Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 4 & -5 \\ -2 & 1 \end{pmatrix} \mathbf{Y}.$$

First let's see what `MatrixFields` tells us about the eigenvalues and eigenvectors of the matrix $\mathbf{A}$.



Aside from the theory of algebraic linear equations

For what matrices $\mathbf{B}$ does the equation $\mathbf{BY} = \mathbf{0}$ have nontrivial solutions?

Singular Matrices. The matrix equation

$$\mathbf{BY} = \mathbf{0}$$

has nontrivial solutions $\mathbf{Y}$ if and only if $\det \mathbf{B} = 0$.

Note: Most matrices are nonsingular.

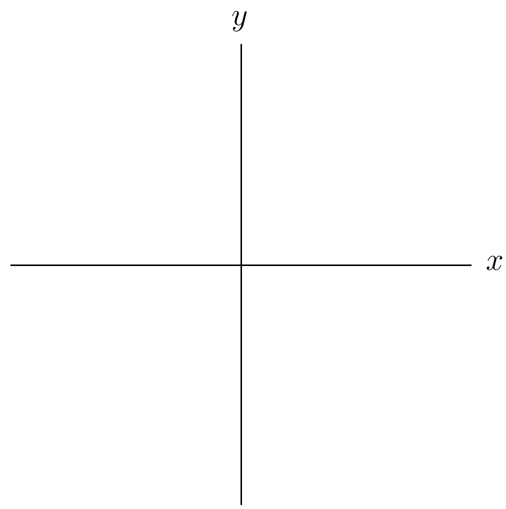
Now let's use this theorem to find eigenvalues and eigenvectors:

Back to the example:

Example. Find the general solution to

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 4 & -5 \\ -2 & 1 \end{pmatrix} \mathbf{Y}.$$

Using `HPGSystemSolver`, we plot the phase portrait for this system.



Facts about eigenvalues and eigenvectors: Given a 2×2 matrix $\mathbf{A}$,

1. The characteristic equation can have two real roots, one real root of multiplicity two, or two complex conjugate roots.

2. Given an eigenvector $\mathbf{Y}_0$ associated to an eigenvalue λ, then any nonzero scalar multiple $\mathbf{Y}_0$ is also an eigenvector associated to λ.

3. Eigenvectors associated to distinct eigenvalues are linearly independent.