Existence and Uniqueness Theory for Systems

There is an existence and uniqueness theorem for systems just like the theorem for equations.

Existence and Uniqueness Theorem. Let

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(t, \mathbf{Y})$$

be a system of differential equations. Suppose that t_0 is an initial time and $\mathbf{Y}_0$ is an initial value. Suppose also that the function $\mathbf{F}$ is continuously differentiable. Then there is an $\epsilon > 0$ and a function $\mathbf{Y}(t)$ defined for $t_0 - \epsilon < t < t_0 + \epsilon$ such that

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(t, \mathbf{Y}(t)) \quad \text{and} \quad \mathbf{Y}(t_0) = \mathbf{Y}_0.$$

In other words, $\mathbf{Y}(t)$ satisfies the initial-value problem. Moreover, for t in this interval, this solution is unique.

There is an important consequence of the Uniqueness Theorem for autonomous systems: Given the autonomous system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y}).$$

Let $\mathbf{Y}_0$ be an initial condition such that $\mathbf{Y}_1(t)$ is a solution that satisfies $\mathbf{Y}(t_1) = \mathbf{Y}_0$ and $\mathbf{Y}_2(t)$ is another solution that satisfies $\mathbf{Y}(t_2) = \mathbf{Y}_0$. Then

$$\mathbf{Y}_2(t) = \mathbf{Y}_1(t - (t_2 - t_1)).$$

Example. Consider the second-order equation

$$\frac{d^2y}{dt^2} + y = 0,$$

which is equivalent to the system

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -y. \end{aligned}$$

Note that

$$\mathbf{Y}_1(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

are both solutions to the system. How are $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ related?

There is an animation on the web site that illustrates this phenomenon.

Here is an informal restatement of this consequence of uniqueness:

> For an autonomous system, if two solution curves in the phase plane touch, then they are identical.

Linear systems

Linear systems and second-order linear equations are the most important systems we study in this course.

What is a linear system with two dependent variables?

What is a second-order linear equation?

How do we write linear systems in vector notation?

Recall two examples that we have already discussed.

Example 1. We have already calculated the general solution to the partially decoupled system

$$\frac{dx}{dt} = 2y - x$$
$$\frac{dy}{dt} = y.$$

It is

$$x(t) = y_0 e^t + (x_0 - y_0)e^{-t}$$
$$y(t) = y_0 e^t.$$

Example 2. For the damped harmonic oscillator

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0,$$

we found two (scalar) solutions

$$y_1(t) = e^{-t} \qquad \text{and} \qquad y_2(t) = e^{-2t}.$$

You should also recall that this second-order equation can be converted to a first-order system where

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -2y - 3v.$$

The vector form of a linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

suggests the use of matrix multiplication:

Given a linear system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y},$$

how do we calculate the vector in the vector field at any given point $\mathbf{Y}_0$?

How do we calculate the equilibrium points of

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}?$$