

Sketching component graphs

Last class we focused on the phase portrait for a sink (two negative and distinct eigenvalues). Once we understand the phase portrait, we should also be able to sketch the component graphs without `HPGSystemSolver`.

Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{Y}.$$

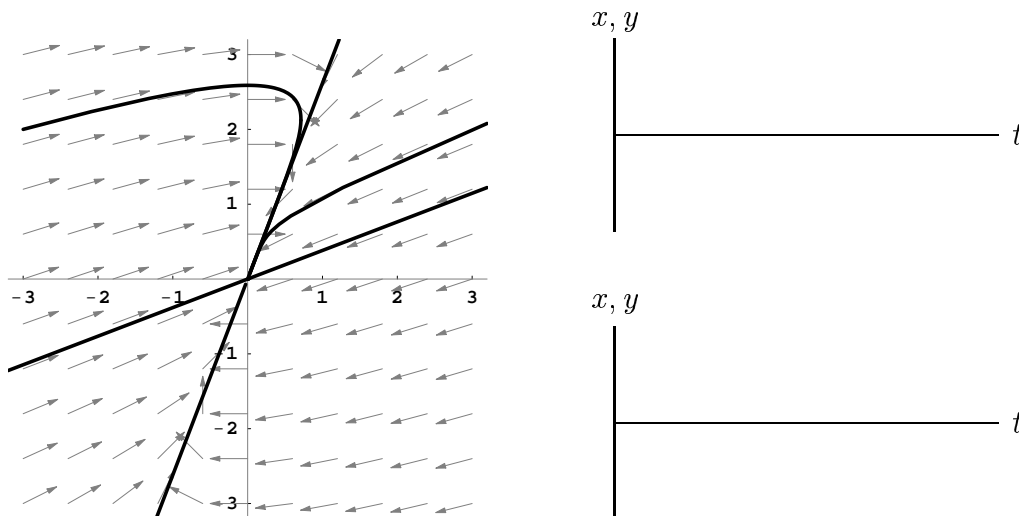
For this example, the eigenvalues are

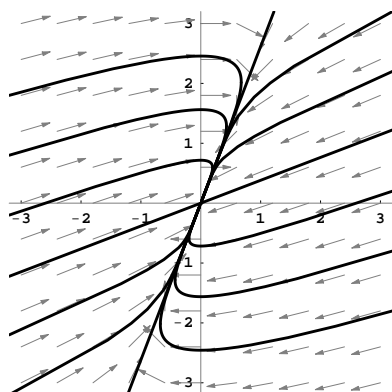
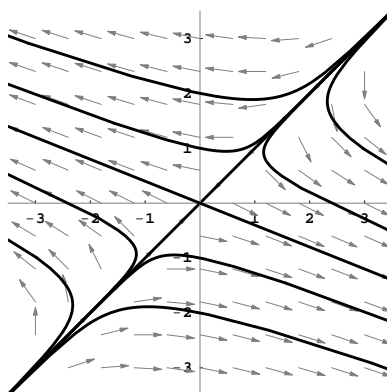
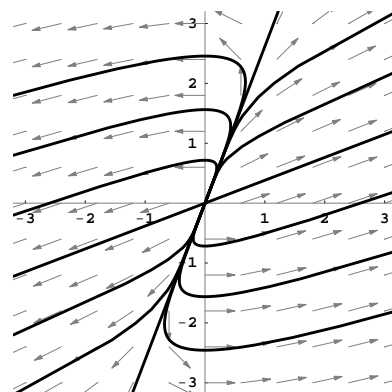
$$\lambda = \frac{-3 \pm \sqrt{5}}{2}.$$

Both are negative.

The eigenline corresponding to the slow eigenvalue $\lambda_2 = \frac{1}{2}(-3 + \sqrt{5})$ has slope that is approximately 2.6, and the eigenline corresponding to the fast eigenvalue $\lambda_1 = \frac{1}{2}(-3 - \sqrt{5})$ has slope that is approximately 0.4.

Now let's sketch the $x(t)$ - and $y(t)$ -graphs that correspond to the initial conditions $(-3, 2)$ and $(3, 2)$.



Summary for real and distinct eigenvaluessink ($\lambda_1 < \lambda_2 < 0$)saddle ($\lambda_1 < 0 < \lambda_2$)source ($0 < \lambda_1 < \lambda_2$)**Complex eigenvalues**

What happens if the eigenvalues of the system are complex numbers?

Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \mathbf{Y}.$$

Let's see that happens if we take a look at this system using `MatrixFields` and then we'll compute the eigenstuff for this matrix.

We now have a complex-valued solution of the form

$$\mathbf{Y}_c(t) = e^{(-2+i)t} \begin{pmatrix} 2 \\ 1+i \end{pmatrix}.$$

We are interested in real-valued solutions. What good is this complex-valued solution?

Once again Euler comes to the rescue: Remember the power series for the exponential function? It is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Let's use this series where $x = bi$.

We will use Euler's formula applied to the complex-valued function

$$e^{(a+bi)t}.$$

But why does this help us solve our differential equation?

Theorem. Consider

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y},$$

where \mathbf{A} is a matrix with real entries. If $\mathbf{Y}_c(t)$ is a complex-valued solution, then both

$$\operatorname{Re}\mathbf{Y}_c(t) \quad \text{and} \quad \operatorname{Im}\mathbf{Y}_c(t)$$

are real-valued solutions, and they are linearly independent.