Linear systems

Last class we started to discuss systems that are linear. They are the ones that can be written in vector form as

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix},$$

where a, b, c, and d are constants. They are also referred to as the coefficients or as the parameters.

Recall two examples that we have already discussed.

Example 1. We have already calculated the general solution to the partially decoupled system

$$\frac{dx}{dt} = 2y - x$$
$$\frac{dy}{dt} = y.$$

It is

$$x(t) = y_0 e^t + (x_0 - y_0)e^{-t}$$
$$y(t) = y_0 e^t.$$

Written in vector notation, this general solution is

$$\mathbf{Y}(t) = e^t \begin{pmatrix} y_0 \\ y_0 \end{pmatrix} + e^{-t} \begin{pmatrix} x_0 - y_0 \\ 0 \end{pmatrix}.$$

Example 2. For the damped harmonic oscillator

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0,$$

we found two (scalar) solutions

$$y_1(t) = e^{-t} \qquad \text{and} \qquad y_2(t) = e^{-2t}.$$

You should also recall that this second-order equation can be converted to a first-order system where

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -2y - 3v. \end{aligned}$$

The vector form of a linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

suggests the use of matrix multiplication:

Given a linear system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y},$$

how do we calculate the vector in the vector field at any given point $\mathbf{Y}_0$?

How do we calculate the equilibrium points of $\dfrac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$?

Theorem. The origin is always an equilibrium point of a linear system. It is the only equilibrium point if and only if $\det \mathbf{A} \neq 0$.

The Linearity Principle

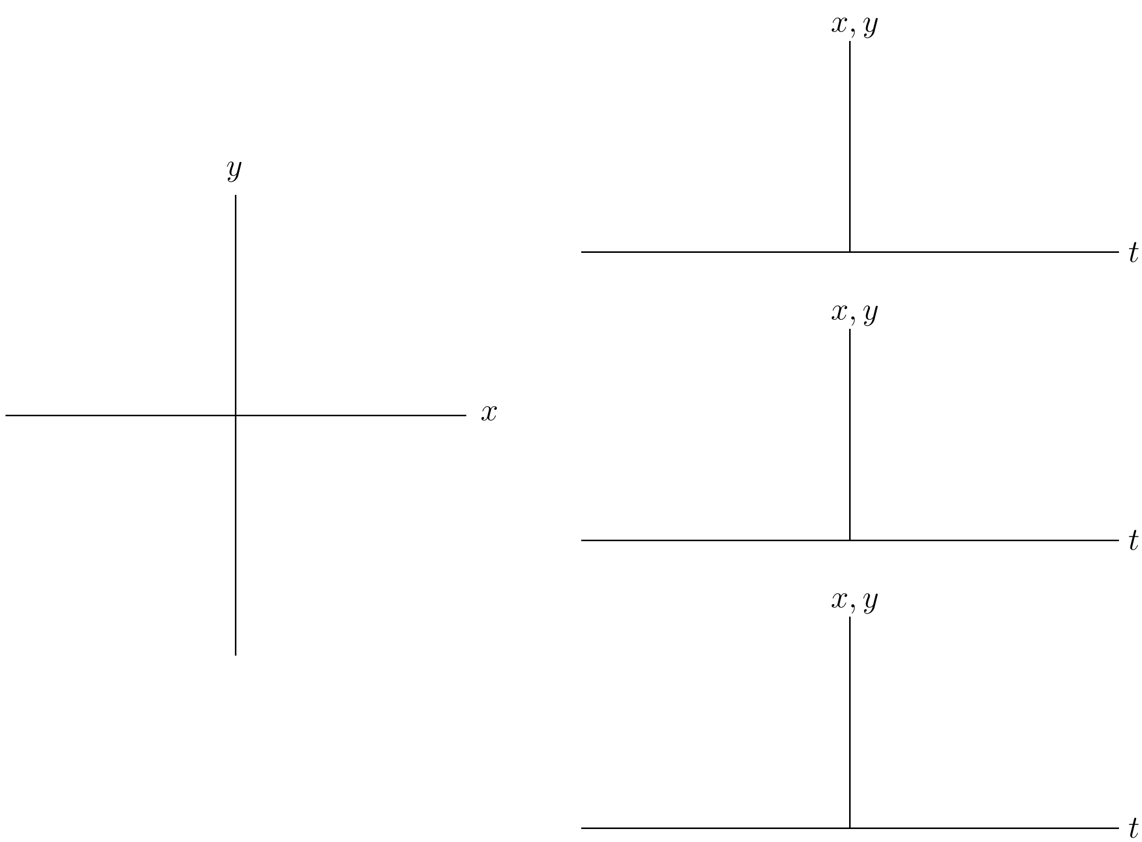
Let's return to Example 1. For practice, I'll use vector notation this time:

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{Y}$$

Also consider three different initial conditions

$$\mathbf{Y}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \mathbf{Y}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \mathbf{Y}_3 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Let's see what happens when we graph the corresponding solutions.



How are these three solutions related?

Linearity Principle Suppose

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

is a linear system of differential equations.

1. If $\mathbf{Y}(t)$ is a solution of this system and k is any constant, then $k\mathbf{Y}(t)$ is also a solution.
2. If $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are two solutions of this system, then $\mathbf{Y}_1(t)+\mathbf{Y}_2(t)$ is also a solution.

Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{Y}$$

and the two solutions

$$\mathbf{Y}_1(t) = \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = \begin{pmatrix} e^{t} \\ e^{t} \end{pmatrix}.$$

Any linear combination of $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ is also a solution to the system.

We can use this observation to solve any initial-value problem involving this system.

Example. Solve

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

In general, how many solutions do we need to be able to solve any initial-value problem?