We will apply what we have learned about linear systems to solve second-order homogeneous linear equations.

Let's return to the guessing technique for second-order equations that we learned about a month ago and see how it relates to what we have done with linear systems recently.

Example. Consider the equation

$$2\frac{d^2y}{dt^2} + 9\frac{dy}{dt} + 4y = 0.$$

1. Use a guessing technique to find two nonzero solutions $y_1(t)$ and $y_2(t)$ that are not multiples of each other.

2. Convert this equation to a first-order system and determine the analogous solutions $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$.

3. In what way are $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ special solutions?

Our goal today is to see how this guessing technique can be used to solve all second-order homogeneous equations.

Consider

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$$

with its characteristic equation

$$a\lambda^2 + b\lambda + c = 0$$

as well as the corresponding system

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -\frac{c}{a}\,y - \frac{b}{a}\,v \end{aligned}$$

with its characteristic equation

$$\det\begin{pmatrix} -\lambda & 1 \\ -\frac{c}{a} & -\frac{b}{a} - \lambda \end{pmatrix} = 0.$$

Useful observation: If λ is an eigenvalue, the vector

$$\mathbf{Y}_0 = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$$

is *always* an associated eigenvector.

Let's see what that tells us about solutions to the second-order equation. There are three cases:

1. Two real, distinct, nonzero eigenvalues λ_1 and λ_2:

2. A complex-conjugate pair of eigenvalues $\lambda = \alpha \pm i\beta$, with $\beta \neq 0$:

3. One nonzero real eigenvalue λ of multiplicity two:

Conclusion: We can determine the general solution of a second-order homogeneous linear equation

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$$

immediately from the characteristic equation

$$a\lambda^2 + b\lambda + c = 0.$$

YOU DO NOT NEED TO CALCULATE THE EIGENVECTORS OR EVEN REDUCE TO A FIRST-ORDER SYSTEM if you simply want to produce the general solution of a second-order linear equation.

Application: We can apply what we have learned to the (damped) harmonic oscillator

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0.$$

In this case, we are assuming that the parameters m and k are positive and that $b \geq 0$. The characteristic equation

$$m\lambda^2 + b\lambda + k = 0$$

has eigenvalues

$$\frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$

There are three cases based on the value of the discriminant $b^2 - 4mk$.

1. $b^2 - 4mk < 0$

2. $b^2 - 4mk = 0$

3. $b^2 - 4mk > 0$

We can see the progression from underdamped to critically damped to overdamped with a Quicktime animation I have posted on the web site.