Last class we did sinusoidal forcing with damping. Today we discuss sinusoidal forcing in the absense of damping.

First, there is one piece of unfinished business from Friday's class. On Friday, we calculated the steady-state solution

$$y_p(t) = -\tfrac{1}{4}(\cos 2t - \sin 2t)$$

for the equation

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + 2y = \cos 2t,$$

and we did so using computations that involved complex numbers. In fact, we found $y_p(t)$ as the real part of

$$y_c(t) = -\tfrac{1}{4}(1+i)\, e^{(2i)t}.$$

At the end of class, I was in the middle of justifying why the complex number

$$a = -\tfrac{1}{4}(1+i)$$

tells us everything we need to know about the steady-state solution.

Using polar coordinates in the complex plane (see pp. 713–715 in Appendix B), we see that

$$a = -\tfrac{1}{4}(1+i) = \tfrac{\sqrt{2}}{4}\, e^{i(-\frac{3}{4}\pi)}.$$

What does this polar representation of a tell us about the steady-state solution?

Sinusoidal forcing without damping

Now for sinusoidal forcing in the absense of damping. For example, consider the mass-spring system without the dashpot.

Example. Let's find the general solution to

$$\frac{d^2y}{dt^2} + 3y = \cos \omega t.$$

Note the absense of a damping term. We want to see what happens with various forcing frequencies.

Unfortunately the parts of the solution that correspond to the associated homogeneous equation do not die out. So to get some qualitative understanding in this case, we make a simplifying assumption. We consider the solution that satisfies the initial condition $(y(0), y'(0)) = (0, 0)$.

On the web site, there is a Quicktime animation of the graphs of these solutions as we vary the forcing frequency ω. The following trig identity helps us interpret what we see in the animation.

Trig identity:

$$\cos at - \cos bt = -2\,(\sin \alpha t)\,(\sin \beta t)$$

where

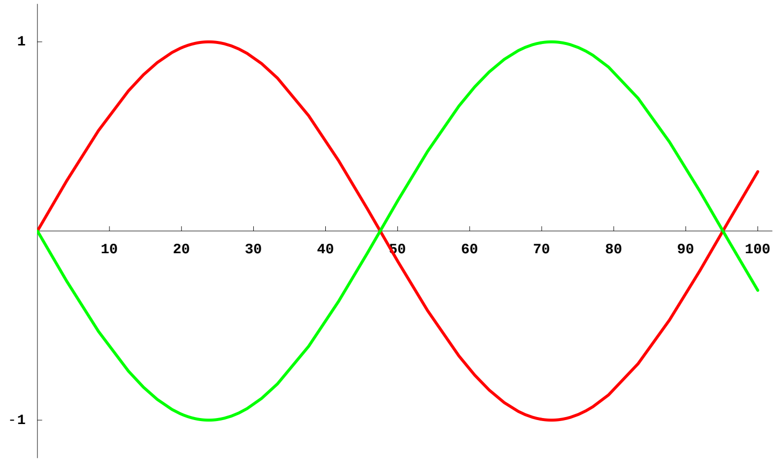
$$\alpha = \frac{a+b}{2} \quad \text{and} \quad \beta = \frac{a-b}{2}.$$

The number α is the average of a and b, and β is called the *half-difference* of a and b.

Example. Let's use this trig identity to get a rough idea of the graph of

$$\cos \omega t - \cos \sqrt{3}\, t$$

where $\omega = 1.6$.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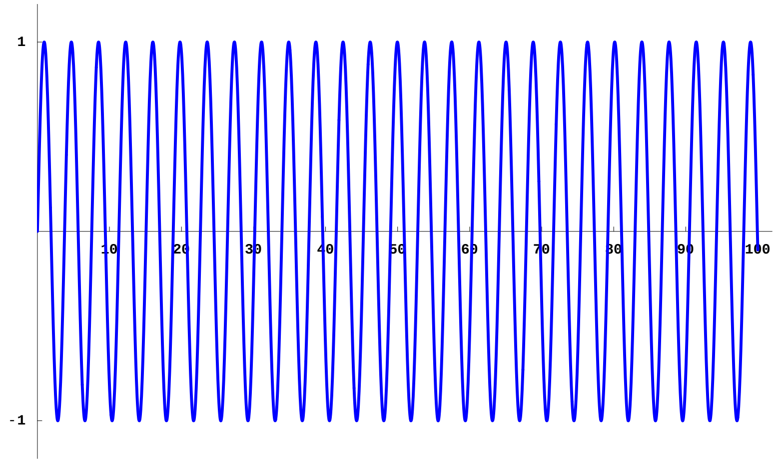
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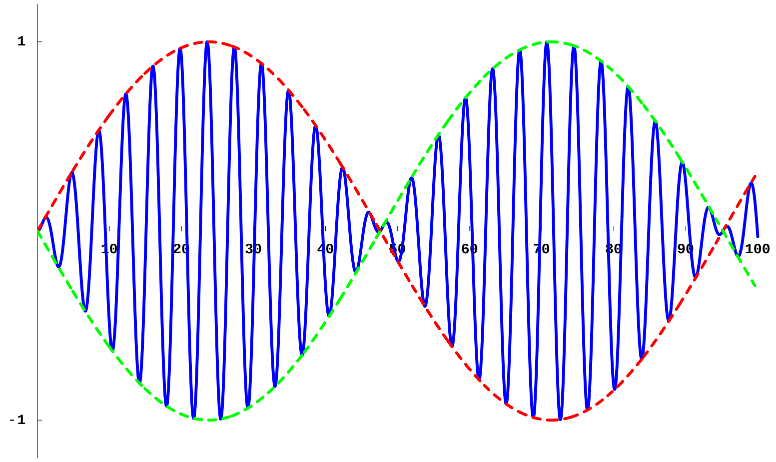
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Let's return to the solution to

$$\frac{d^2y}{dt^2} + 3y = \cos \omega t$$

that satisfies the initial condition $(y(0), y'(0)) = (0, 0)$. If $\omega \neq \pm\sqrt{3}$, the solution is

$$y(t) = \frac{1}{3 - \omega^2}(\cos \omega t - \cos \sqrt{3}\, t).$$

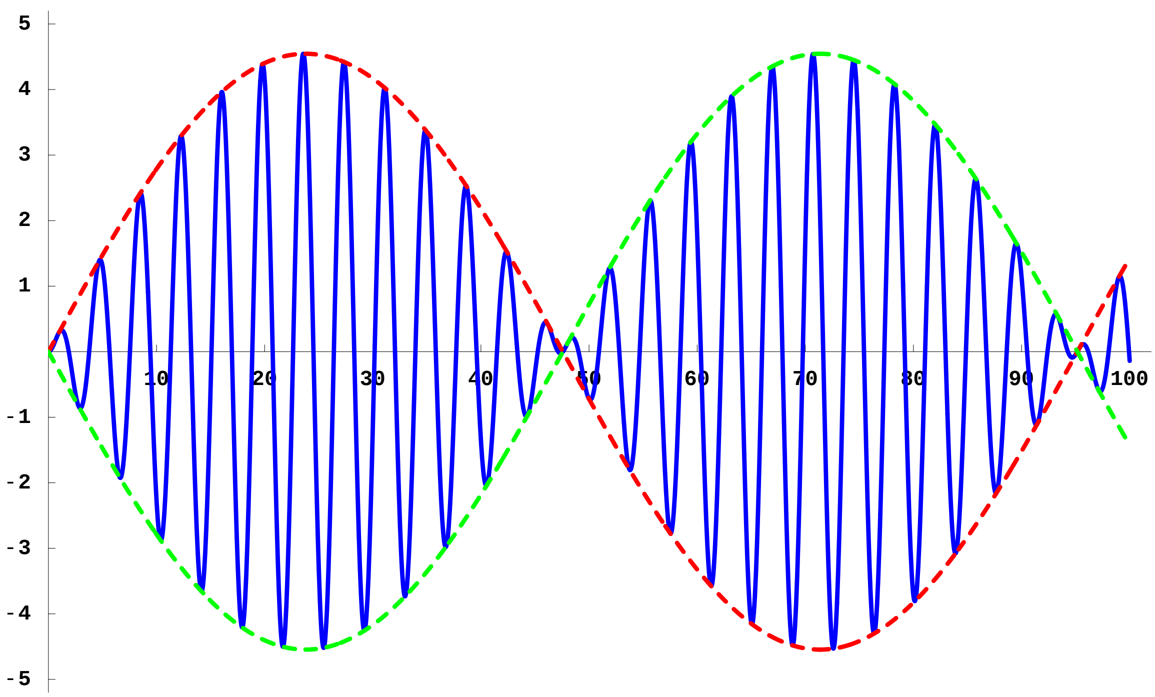
Applying the trig identity, we obtain

$$y(t) = \frac{-2}{3 - \omega^2}\ (\sin \alpha t)\ (\sin \beta t)$$

where

$$\alpha = \frac{\omega + \sqrt{3}}{2} \quad \text{and} \quad \beta = \frac{\omega - \sqrt{3}}{2}.$$

Here is the graph of this solution in the case where $\omega = 1.6$.



What happens if $\omega = \sqrt{3}$?